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Wave Propagation Across Ion Cyclotron Resonance Harmonic Layers

Kaya Imre and Harold Weitzner

U.S. Department of Energy

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WAVE PROPAGATION ACROSS ION CYCLOTRON RESONANCE HARMONIC LAYERS

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ABSTRACT

Wave propagation across ion cyclotron resonance layers is investigated by use of a boundary layer analysis for a perpendicularly stratified, weakly inhomogeneous plasma. Corresponding to the situation in many plasmas of interest, it is assumed that the wave frequency is much less than the ion plasma frequency, but that the ratio of ion plasma frequency to wave frequency times the ratio of ion thermal speed to the speed of light is small. Non-trivial effects of the cyclotron resonance appear in the boundary layer and modify the current density from the cold plasma limit. These corrections suggest the breakdown of the geometrical optics limit. Wave equations for the field valid within an arbitrary harmonic resonance layer are derived, as well as for the two ion hybrid resonance case. Analytical proofs of reciprocity relations are given for the general wave propagation problems. For fundamental resonance, second harmonic resonance, and ion-ion hybrid resonance proofs of the properties that (i) the transmission coefficient remains the same as its geometrical optics value, and (ii) the absence of reflection of the fast mode incident from the high field side are given. Numerical results for the cases of fundamental, second, and third ion resonances, and for the ion-ion hybrid resonance are presented.

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I. INTRODUCTION

Heating of plasmas by waves at frequencies matching some ion cyclotron harmonic resonance has attracted attention both experimentally and theoretically. In particular, the second harmonic resonance heating, with or without minority ion species at the fundamental resonance, has been used experimentally with success to heat plasmas. Using frequencies matching a higher ion cyclotron harmonic resonance may be attractive since smaller antennas for launching the waves are appropriate. Successful use of these heating schemes depends on the energy absorption efficiency, as well as the accessibility of the incident waves to the resonance region. The purpose of this paper is to present a study of propagation, absorption, and mode conversion of waves across cyclotron harmonic resonances for a perpendicularly stratified medium for waves propagating nearly perpendicularly. When the waves propagate nearly perpendicular to the equilibrium magnetic field, the standard geometrical optics approximation is expected to breakdown near the resonance layer. Associated with this failure, reflection and mode coupling effects may also become significant, so that a full wave analysis is required. In Refs.1-4 we have studied similar problems for the electron cyclotron harmonics. By use of a boundary layer expansion about the resonance surface we were able to obtain full wave solutions.

The work here overlaps some of the extensive analysis of Swanson,[5,6] and Colestock and Kashuba[7], but it differs in several aspects. We prefer to treat the system by boundary layer expansions rather than inference of wave equations from uniform medium dispersion relations and energy conservation relations. We believe that the boundary layer expansion technique gives the cleanest and simplest set of approximate equations for the study of wave propagation near a cyclotron resonance. By evaluating the current from the Vlasov equation with our method, we obtain directly the wave equation for

the electromagnetic wave field, and not more abstract wave equations. Since we are finally interested in complete calculations of full wave fields, see e.g. Refs 1-3, rather than less complete but far simpler variational calculations, we focus on a full wave treatment. By the use of full wave equations, we obtain a group of exact properties of the solutions. Specifically, we obtain properties that are independent of the equilibrium distribution function, provided the distribution function is isotropic in velocity space. We obtain reciprocity theorems and we find general connections between the full wave solution to a scattering problem, and the geometrical optics solution to the same problem.

Recently, one of us (K.I.) has investigated the wave propagation across the higher (i.e., third and higher) electron cyclotron harmonic layers using the same method.[4] It was shown that the relativistic corrections wash out resonance effects, and geometrical optics approach becomes applicable. In that case, to the lowest significant order in the expansion parameter, the ratio of the electron thermal speed to the speed of light, we found that the cold plasma approximation holds, and the effects of the higher harmonic resonance are negligible. In the present work we address ourselves to similar questions in the case of ion resonances. By making use of a similar boundary layer analysis, we show that, unlike the electron case, there exists certain parameter domains where the geometrical optics approximation fails and full wave treatments are needed. For the case of ions we expect relativistic corrections to be negligible, but as a matter of completeness we determine the extremely small range in $n_{||}$ in which they might appear.

We consider in detail three essentially distinct, but closely related problems, Nth harmonic resonance $N \geq 2$, fundamental cyclotron resonance, and ion-ion hybrid resonance with a majority ion species at second harmonic res-

onance and minority species at fundamental resonance. We obtain wave equations appropriate to each case. In all these problems we have a fast mode — an extraordinary mode for $N \geq 2$ and a magnetosonic mode for $N=1$ — which propagates on both sides of resonance, and a slow mode — a Bernstein mode for $N \geq 2$ — which propagates on only one side of resonance.[8] We characterize the wave propagation problem in terms of coefficients of wave transmission, mode converted transmission, reflection, and mode converted reflection. We prove a series of exact results corresponding to the generalization of the reciprocity theorems of Chiu and Mau.[9] In another vein, for the cases of second harmonic resonance, fundamental resonance, and ion-ion hybrid resonance we obtain exact formulas for the transmission coefficient for incident fast modes, and show that for a fast mode incident from the high field side no fast mode is reflected. We believe that our method of proof of these latter results would also extend directly to higher harmonic resonances. Additionally, for purely perpendicular propagation for all cases except ion-ion hybrid resonance we obtain integral representations of the solutions. Finally, we examine these cases numerically to obtain information on plasmas of some laboratory interest.

The organization of this paper is as follows. Section II contains the formulation of the boundary layer analysis, and the derivation of the current density from the Vlasov equation for Nth ion cyclotron harmonic resonance. We use the current density with Maxwell's equations to derive field equations valid within the boundary layer in Sec.III for the case of Nth harmonic resonance $N \geq 2$. We also establish the connection of the solutions of these equations with those given by geometrical optics. Furthermore, the wave equations enables us to prove the reciprocity relations in general terms. These relations state that (i) the transmission coefficient for the X-mode is independent of the direction of incidence, (ii) the reflected mode

coupling coefficients are the same in the cases of fast mode and the slow mode incidences, and (iii) the transmitted mode conversion coefficients are the same in the cases of fast and slow mode incidences. The validity of these relations does not depend explicitly either on parallel index of refraction or on the equilibrium distribution function. The method of proof of Chiu and Mau' which leads to similar reciprocity relations cannot be directly applied in our case as it exploits the symmetry of the full system which may not be retained in the approximate equations derived for the resonance layer. Two other results found in the special case of zero parallel refractive index are that the reflection coefficient is zero for a high field side incident fast mode, and that the transmission coefficient for this mode is the same as that predicted by geometrical optics. We also show that these results hold generally for second harmonic resonance. We suspect the results are true for all harmonics, and the proof would be an extension of the theorem in the Appendix.

In Sec.IV, we study the problem of fundamental resonance with one ion species. We show that there a thin boundary layer about the fundamental resonance layer in which geometrical optics fails. The dispersion relation in a uniform medium predicts a bifurcation point within this resonance layer, where the fast and slow branches of the magnetosonic mode coalesce. Partially on the basis of the cold plasma model, one expects that a fast mode is largely unaffected by the fundamental cyclotron resonance, and that the slow mode may be absorbed. The Vlasov model is more complex, with mode conversion and tunnelling occurring. Nonetheless, this crude picture persists. The mode coupling rapidly vanishes outside a narrow cone of incident waves. The fast mode incident from the low field side is also partially reflected. The proof of the reciprocity relations for this case is also

given. The fast mode incident transmission coefficient is given exactly, and the high field side incident fast mode has no reflected fast mode.

It is well-known that the absorption of waves propagating across the second harmonic layer is significantly improved by the inclusion of a small amount of another ion species at the fundamental resonance within the same resonance surface. In Sec.V, we study this problem by our boundary layer technique. Clearly, when the transmission coefficient is near unity, the absorption is necessarily poor. We show that by inclusion of minority ions we can significantly reduce the transmission coefficient. A substantial fraction of the incoming wave energy is then, in fact, absorbed by ions. The addition of the minority species even further reduces the parameter range in which geometrical optics can be applied across resonance. In fact in some cases the reflected mode coupling remains significantly large even when the parallel refractive index is increased close to its left-hand cut-off value. It is largely because of this property that the absorption of the waves incident from the low field side is higher than that of the high field side. Again, reciprocity theorems and exact values of transmission coefficient are given. In Sec.VI, we present our numerical results for the various cases studied in this paper. We show that the third harmonic resonance can be used to heat tokamak-like plasmas by low field side incident fast waves. The high field side incident fast (X) or slow (Bernstein) modes may provide acceptable absorption rates if the parallel refractive index is sufficiently large. Absorption of fast magnetosonic waves at the fundamental resonance could be significant for dense and/or large plasmas, whereas the slow mode is readily absorbed even when $n_{||}$ is quite small. We study the parameter range corresponding to PLT, JET, and CIT devices for the case of ion-ion hybrid resonance. We find that when the minority number density is approximately 1% of the electron number density the low field side incident

fast mode absorption rate peaks strongly. When $n_{||}$ is sufficiently small most of the absorbed incident energy is deposited to the minority species. We find that in the case of CIT, absorption of the high field side incident fast or slow modes is quite negligible.

Section VII has a brief discussion of the results and concluding remarks. An appendix contains a useful theorem on asymptotic expansions of a solution of ordinary differential equations.

II. EVALUATION OF CURRENT DENSITY

A. The Vlasov Equation within the boundary layer

Within the ion cyclotron resonance layers we represent the plasma ions by the linearized Vlasov equation, and we solve the Vlasov equation by perturbation expansion. The consistency of the expansion is not fully demonstrated until the next section. The equilibrium magnetic field $\mathbf{B}_0$ is assumed to be along the z direction, and its magnitude varies slowly along the x direction. The time and z dependences are assumed to be harmonic in the form $\exp[-i\omega(t - zn_{||}\,\omega/c)]$. We normalize the velocity variable with respect to the ion thermal speed v_i, and introduce

$$\begin{aligned} \mathbf{u} &= v_i\, \mathbf{w}, \\ g\, d^3w &= e\, v_i\, f_i\, d^3u, \\ n_i\, G\, d^3w &= F_i\, d^3u, \end{aligned} \tag{1}$$

where f_i and F_i denote perturbed, and equilibrium ion distribution functions, respectively. We assume that G is isotropic in the velocity space. Within the Nth harmonic resonance layer we introduce the stretched spatial variable

$$\xi = \frac{1}{\delta}\left[\frac{1}{N} - \frac{\Omega(x/L)}{\omega}\right], \tag{2}$$

where the value of δ will be determined to assure consistency of the expansion, and $\omega = N\Omega(0)$. We write the ion Vlasov equation in the form

$$(1 - \frac{j}{N} + \delta\,\Delta_j)g_j + \mathcal{L}(g_{j+1} + g_{j-1}) = b_j, \tag{3}$$

where we have expanded g in a Fourier series in gyrophase angle ϕ, and we set

$$\begin{aligned}
&w = (w_\perp \cos\phi,\ w_\perp \sin\phi,\ w_\parallel),\\
&\Delta_j = j\xi - \eta\delta^{-1} n_\parallel w_\parallel + \eta^2\delta^{-1}\frac{w^2}{1+\gamma},\\
&\mathcal{L} = i\frac{\eta}{2} w_\perp \frac{c}{\omega}\frac{\partial}{\partial x} = i\frac{\varepsilon}{2} w_\perp K(\delta\xi)\frac{\partial}{\partial\xi},\\
&b_j = \frac{\omega}{4\pi i}\gamma X_i[\frac{1}{2}\frac{\partial G}{\partial w_\perp}(\delta_j^{-1}E_- + \delta_j^{+1}E_+) + \frac{\partial G}{\partial w_\parallel}\delta_j^0 E_\parallel],\\
&\Omega = Z_i eB_0(x/L)/m_i c,\\
&\gamma = (1 + \eta^2 w^2)^{1/2}\\
&\eta = v_i/c,\\
&\varepsilon = \eta n_c,\\
&E_\mp = E_x \mp iE_y,\ E_\parallel = E_z,
\end{aligned} \tag{4}$$

where n_c is the cold plasma refractive index evaluated at the resonance. We specify n_c in the next section. In order to evaluate the current in terms of the field variables, we must calculate g_{-j}, $j = 0, \mp 1$ from the system (3). In the present work we are only interested in the lowest significant order contributions of the resonance effects. Thus we keep only the terms of order zero, namely

$$g_j^{(0)} = g_j\,\{1 + 0(\varepsilon^2)\}, \tag{5}$$

and suppress the superscript (0) for brevity, so that (12) can be written as

$$(1 - \frac{j}{N})g_j + \mathcal{L}(g_{j+1} + g_{j-1}) = b_j, \quad (j \neq N),$$
$$\delta\ \Delta_N\ g_N + \mathcal{L}(g_{N+1} + g_{N-1}) = b_N, \tag{6}$$

with $K(\delta\xi)$ replaced by $K(0)$ in $\mathcal{L}$.

The right hand side of the equation corresponding to the resonant component g_N is zero except for the case N=1. Thus we must treat the case of the fundamental resonance separately. We first consider the case when $N \geq 2$.

B. Ion Current when $N \geq 2$

We avoid making explicit assumptions about the relative orders of the Fourier components (the ordering will become evident at the end of our analysis), except that we hypothesize that g_j tends to zero for large $|j|$. By induction we obtain the relation

$$g_{-2} = -\frac{N}{N+1}\mathcal{L}g_{-1} + O(g_{-M}), \quad M>3, \tag{7}$$

from which it follows that we may drop the term involving g_{-2} in the equation corresponding to $j = -1$ in (6). Next, we express g_2 in terms of g_1 and g_N, by making use of (6) for $j=2,3,\cdots,N-1$, yielding

$$g_2 = -\frac{N}{N-2}\mathcal{L}g_1 + \frac{(-1)^N}{(N-2)!}N^{N-2}\mathcal{L}^{N-2}g_N. \tag{8}$$

We may relate g_N to g_1 by

$$\delta\ \Delta_N\ g_N = \frac{N^{N-2}}{(N-2)!}[(-1)^{N+1}\mathcal{L}^{N-1}g_1 + \mathcal{L}^{N-2}g_{2N-1}]. \tag{9}$$

We complete our truncation process by expressing g_{2N-1} in terms of g_N. We accomplish this by using the relation

$$g_{m+N} = \frac{N}{m}\mathcal{L}g_{m+N-1} + O(\epsilon^M g_{m+N+M}), \quad m>1, \ M>0, \tag{10}$$

which again may be established by induction. For $m = N-1$, and for M sufficiently large so that $\epsilon^M g_{2N+1+M} = O(\epsilon^2)$, the latter gives

$$g_{2N-1} = \frac{N^{N-1}}{(N-1)!} \mathcal{L}^{N-1} g_N. \tag{11}$$

We now obtain the solution of the Vlasov equation by using the relations (7), (8), (9), and (11). By making use of (6) with j=0, j=-1, and j=1, we express g_1 in terms of g_2, and thus in terms of g_N, giving

$$g_1 = \frac{(-N)^{N-1}}{(N-1)!} \mathcal{L}^{N-1} g_N + \bar{b}, \tag{12}$$

where

$$\bar{b} = \frac{N}{N-1}(b_1 - \mathcal{L}\, b_0 + \frac{N}{N+1} \mathcal{L}^2\, b_{-1}). \tag{13}$$

We next substitute (12) into (9), and use (11) to obtain

$$\delta\, \Delta_N\, g_N = - \frac{(-N)^{N-2}}{(N-2)!} \mathcal{L}^{N-1}\, \bar{b}. \tag{14}$$

We have thus expressed the resonant component in terms of the field variables alone, and thus we can evaluate every g_j in terms of b_j. In particular,

$$g_1 = \bar{b} + \frac{N^{2N-3}}{(N-1)!\ (N-2)!}\ \mathcal{L}^{N-1}\ \frac{1}{\delta\ \Delta_N}\ \mathcal{L}^{N-1}\ \bar{b}\ . \tag{15}$$

The second term in the right-hand side of (15) represents the contribution of the Nth harmonic resonance to the lowest significant order in the expansion parameter. Similar relations for g_0 and g_{-1} are also obtained. The corrections are of higher order in ϵ in these components:

$$g_0 = b_0 - \frac{N}{N-1} \mathcal{L}\, b_1 - \frac{N}{N+1} \mathcal{L}\, b_{-1} + \frac{N^{2N-2}}{[(N-1)!]^2} \mathcal{L}^N \frac{1}{\delta\ \Delta_N} \mathcal{L}^{N-1}\, b_1,$$
$$g_{-1} = \frac{N}{N+1}(b_{-1} - \mathcal{L}\, b_0 + \frac{N}{N-1} \mathcal{L}^2\, b_1) + \frac{N^{2N}}{(N+1)!\ (N-1)!} \mathcal{L}^{N+1} \frac{1}{\delta\ \Delta_N} \mathcal{L}^{N-1}\, b_1. \tag{16}$$

We obtain for the ion current density from

$$J_{\mp}^{ion} \equiv \int \frac{d^3w}{\gamma} w_{\perp} g_{\mp 1}, \quad J_{\|}^{ion} = \int \frac{d^3w}{\gamma} w_{\|} g_0, \tag{17}$$

so that

$$\frac{4\pi i}{\omega} J_{\|} = -X_i E_{\|}$$

$$+ 2ic_N X_i \varepsilon D^N (a_{N-1,1} D^{N-1} E_+ - \tfrac{1}{4}\varepsilon^2 a_{N,0} D^{N+1} E_- - i\varepsilon a_{N-1,2} D^N E_{\|}),$$

$$\frac{4\pi i}{\omega} J_- = -\frac{N}{N+1} X_i (E_- - \frac{N}{N+1} \tfrac{1}{2}\varepsilon^2 \langle w_{\perp}^2 \rangle D^2 E_+)$$

$$- c_N X_i \tfrac{1}{4}\varepsilon^2 D^{N+1} (a_{N,0} D^{N-1} E_+ + \tfrac{1}{4}\varepsilon^2 \frac{N(N+2)}{(N+1)^2} a_{N+2,0} D^{N+1} E_- - i\varepsilon a_{N,1} D^N E_{\|}), \tag{18}$$

$$\frac{4\pi i}{\omega} J_+ = -\frac{N}{N-1} X_i (E_+ - \frac{N}{N+1} \tfrac{1}{2}\varepsilon^2 \langle w_{\perp}^2 \rangle D^2 E_-)$$

$$+ c_N X_i D^{N-1} (a_{N-1,0} D^{N-1} E_+ - \tfrac{1}{4}\varepsilon^2 a_{N,0} D^{N+1} E_- - i\varepsilon a_{N-1,1} D^N E_{\|}),$$

where $\langle \cdots \rangle$ denotes velocity space average with respect to the equilibrium distribution function, and

$$c_N = \frac{(-1)^N N (N/2)^{2N-2}}{[(N-1)!]^2} \frac{\varepsilon^{2N-2}}{\delta} \tag{19}$$

$$D = K(0) \frac{d}{d\xi} .$$

The functions $a_{n,m}$ are defined by

$$a_{n,m} = -\frac{1}{2n+2} \int \frac{d^3w}{\Delta_N} w_{\perp}^{2n+1} w_{\|}^m \frac{\partial G}{\partial w_{\perp}}. \tag{20}$$

We now return to the case omitted in this section, N = 1.

C. Fundamental ion resonance

The fundamental ion cyclotron resonance problem differs significantly from the other harmonic resonances which we have just studied. The basic difference stems from the fact that now the resonant cold plasma dielectric tensor element ε_{++} becomes unbounded as one approaches the fundamental resonance. Thus, it becomes possible that E_+ decreases near resonance, and cor-

respondingly the sole effect of resonance may be to shield out E_+. Additionally, although no ion Bernstein modes propagate near the fundamental resonance layer, other modes do exist, and it is necessary to investigate carefully the uniform medium dispersion relation in detail near resonance in order to discover what other modes are present. We give this analysis as part of Sec.IV.

We return to the system (1)-(4), with N = 1, and we take n_c to be the cold plasma refractive index evaluated at the resonance, which we specify later. Again, we solve the chain equations (3) order by order in terms of field variables. To the lowest significant order in ε we obtain

$$g_{-2} = -\frac{1}{3}\mathcal{L}g_{-1},$$
$$g_2 = \mathcal{L}g_1,$$

so that

$$\begin{aligned} \delta\Delta_1 g_1 &= b_1 - \mathcal{L}\, b_0 + \frac{1}{2}\mathcal{L}^2 b_{-1}, \\ g_0 &= b_0 - \frac{1}{2}\mathcal{L}\, b_{-1} - \mathcal{L}\, g_1, \\ g_{-1} &= \frac{1}{2}\, b_{-1} - \frac{1}{2}\mathcal{L}\, b_0 + \frac{1}{2}\mathcal{L}^2 g_1. \end{aligned} \tag{21}$$

For any cyclotron resonance but fundamental we could determine the relation between δ and ε by demanding that current densities of order one; now we must turn to Maxwell's equations to find the relation. We calculate the ion current density without making any assumption on parameters and on the relative ordering of the variables, other than requiring that (21) be correct to leading order. The components of the ion current density are then obtained using (18). We find

$$\begin{aligned} \frac{4\pi i}{\omega} J_+ &= -\frac{1}{\delta} X_i a_{0,0} E_+ + \frac{\varepsilon^2}{4\delta} X_i a_{1,0} D^2 E_- + i\frac{\varepsilon}{\delta} X_i a_{1,1} DE_{||}, \\ \frac{4\pi i}{\omega} J_- &= D^2\frac{\varepsilon^2}{4\delta} X_i a_{1,0} E_+ + X_i(-\frac{1}{2} - \frac{3\varepsilon^4}{64\delta} D^2 a_{2,0} D^2) E_- - \frac{i\varepsilon^2}{4\delta} X_i D^2 a_{1,1} DE_{||}, \end{aligned} \tag{22}$$

$$\frac{4\pi i}{\omega} J_{\|} = \frac{i\varepsilon}{2\delta} X_i Da_{0,1} E_+ - \frac{i\varepsilon^2}{8\delta} X_i Da_{1,1} D^2 E_- + X_i(-1 + \frac{\varepsilon^2}{2\delta} Da_{0,2} D) E_{\|} ,$$

where $a_{n,m}$ is as given in (20) with N = 1.

III. HARMONIC RESONANCE WITH N $\geq$ 2

A. Field Equations within the Resonance Layer

We calculate electron current evaluated at the resonance from the cold plasma conductivity and the additional assumption of charge neutrality of the system, i.e.

$$\frac{4\pi i}{\omega} J_{\mp}^{elec} = \pm N X_i \tag{23}$$

and substitute it with the ion current density (18) into Maxwell's equations. We normalize the problem by assuming that $E_{\mp}$=O(1), and find that the stretching parameter δ must be chosen to be ε^{2N-2} so that the resonance effects will be present at the lowest significant level in the boundary layer expansion. When we drop the terms of order m_e/m_i, and eliminate E_- within the resulting equations we obtain the following equation for E_+ which is valid in the Nth harmonic ion resonance layer

$$[1 + d^2 + (1 + d^2 + \mu)d^{N-1} A\, d^{N-1}]E_+ = 0, \tag{24}$$

where d = d/ds, and

$$\begin{aligned}
&s = \xi \frac{n_c}{K(0)} \frac{\omega}{\omega_{pi}} = -\xi n_c \delta N L\omega/c, \\
&n_c^2 = (\varepsilon_{--} - n_{\|}^2)(\varepsilon_{++} - n_{\|}^2)/(\varepsilon_{xx} - n_{\|}^2), \ \varepsilon_{xx} = \frac{1}{2}(\varepsilon_{++} + \varepsilon_{--}), \\
&\varepsilon_{++} = 1 - \frac{N^2}{N-1} X_i, \quad \varepsilon_{--} = 1 + \frac{N^2}{N+1} X_i, \\
&\mu = (\varepsilon_{--} - n_{\|}^2)/(\varepsilon_{++} - n_{\|}^2), \\
&A = \frac{X_i}{(\varepsilon_{xx} - n_{\|}^2)} \left[\frac{n_c^2}{X_i}\right]^{N-1} \frac{(-1)^{N+1}(N/2)^{2N-1}}{[(N-1)!]^2} a_{N-1,0} .
\end{aligned} \tag{25}$$

The components ε_{ij} given above correspond to the cold plasma dielectric tensor elements evaluated at the resonance, and n_c is the Appleton-Hartree index of refraction again evaluated at the resonance. We note that if $n_{||}^2 \ll X_i$ then we can drop $n_{||}$ in the definitions given in (25). However, we see from (4) that even when these corrections may be dropped, when

$$n_{||}^2 = O(\delta/\eta),$$

the effects of $n_{||}$ on wave propagation are significant. It is remarkable that the relativistic corrections are also significant within the layer provided that the more stringent condition hold that η^2/δ be of order unity or less. However, these corrections are not large enough to suppress the resonance effects as they do in the case of higher electron harmonics, as was demonstrated in Ref.4. Hence the relativistic effects are significant within a thin propagation cone if the following ordering holds:

$$X_i = O(\eta^{-\frac{2N-4}{N-1}}), \quad \text{and} \quad n_{||} = O(\eta).$$

To show that the asymptotic solutions of Eq.(24) can be connected to the geometrical optics solutions, we next derive the asymptotic expansions of the solutions.

B. Asymptotic Expansion of the Solutions of Inner Equation

For $|\xi|$ large, we can write

$$a_{N-1,0} = \frac{1}{N\xi} \langle w_\perp^{2N-2} \rangle, \tag{26}$$

where $\langle \cdots \rangle$ denotes velocity space average with respect to the equilibrium distribution function G, so that Eq.(24) becomes

$$\{1 + d^2 - [b + a(1 + d^2)]d^{N-1} \frac{1}{s} d^{N-1}\} E_+ = 0, \tag{27}$$

where

$$b = \mu a,$$

$$a = \frac{X_i}{(\varepsilon_{xx} - n_{||}^2)} \left[\frac{n_c^2}{X_i}\right]^{N-1} \frac{(-1)^{N+1}(N/2)^{2N-1}}{[(N-1)!]^2} \langle w_{\perp}^{2N-2} \rangle \, (n_c \, \delta \, L \, \frac{\omega}{c}). \tag{28}$$

We note that (27) also represents the case of purely perpendicular propagation in the nonrelativistic limit. For X_i large, the case of interest, $\mu<0$, so that a and b are of opposite sign, and

$$a = |a| \, (-1)^{N+1}, \text{ and } b = |b| \, (-1)^N. \tag{29}$$

We can write the solution of (27) in the form

$$n = -\int e^{ps} \, p^{N-1} \, (a + \frac{b}{1+p^2}) \, e^{P(p)} \, dp, \tag{30}$$

where the contonr of integration in (30) is to be specified later, and

$$P(p) = - a \, \frac{p^{2N-1}}{2N-1} - |b| \sum_{j=0}^{N-2} (-1)^j \, \frac{p^{2j+1}}{2j+1} + \frac{|b|}{2i} \log \frac{p-i}{p+i}$$

The integrand of (30) tends to zero as $|p| \to \infty$ within the sectors

$$\frac{2k-1}{2N-1} \frac{\pi}{2} < \arg p < \frac{2k+1}{2N-1} \frac{\pi}{2}, \text{ for } k+N \text{ odd integer.}$$

The sectors corresponding to $k = \mp(N-1)$ are of particular importance. The contours extending to infinity within these sectors will be seen to include the contributions of the propagating ion Bernstein waves. For $|s|$ large, only the contributions of the branch points at $p_b = \mp i$, and the stationary points p_s need be considered. The stationary points are

$$p_s = (\frac{s}{a})^{\frac{1}{2N-2}} z_k, \quad k=0,1,\cdots,2N-1, \tag{31}$$

where

$$z_k = \exp(i\pi \frac{k}{N-1}), \text{ for } s/a >> 1,$$
$$= \exp(i\pi \frac{2k-1}{2N-2}), \text{ for } s/a << -1.$$

The contributions of the stationary points generate either asymptotically small or asymptotically large solutions except for the branches $p_s = \mp i|p_s|$, which occurs only for $s>0$ (high field side). We find that there are only three (out of 2N) linearly independent solutions which are physically relevant for the scattering problems. The contours C_1, C_2, and C_3 which generate these solutions are shown in Fig.1. The other linearly independent solutions all are asymptotically unbounded as $s \to +\infty$ or $s \to -\infty$. Moreover, these solutions are linearly independent.

By using standard methods we find that the asymptotic expansion of the three linearly independent solutions for $|s|$ large are

$$\begin{aligned} u_{\mp} &= p_{\mp}\, e^{|b|5\pi/4} + q_{\mp}\,, \quad \text{HFS}, \\ &= p_{\mp}\, e^{|b|3\pi/4}, \quad \text{LFS}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} u_3 &= e^{|b|3\pi/4}(e^{|b|\pi/2} - e^{-|b|\pi/2})(p_+ + p_-) + q_+ + q_-, \quad \text{HFS}, \\ &= 0, \quad \text{LFS}, \end{aligned} \tag{33}$$

where HFS and LFS stand for high-field side ($s>0$) and low-field side ($s<0$), respectively, and

$$\begin{aligned} p_{\mp} &= \mp \frac{b}{2}\,\Gamma(\pm i|b|/2)\, \exp[\mp i(\pi\frac{N}{2} + s_1 + \frac{|b|}{2}\log 2|s_2|)], \\ s_1 &= s - \frac{|a|}{2N-1} - |b| \sum_{j=0}^{N-1} \frac{1}{2j+1}, \\ s_2 &= s - |a| - (N-1)|b|, \\ q_{\mp} &= \left|\frac{\pi\, a}{N-1}\, p_s\right|^{1/2} e^{|b|\pi}\, \exp\{\mp i[s|p_s|\frac{2N-2}{2N-1} - \frac{\pi}{4}(2N+1)]\}. \end{aligned} \tag{34}$$

We have

$$|q_{\mp}/p_{\mp}|^2 = e^{2|b|\pi}(e^{|b|\pi/2} - e^{-|b|\pi/2})\,\frac{|p_s|}{\mu(N-1)}\,. \tag{35}$$

In closing this section we note that since $q_{\mp}$ represent incoming and outgoing ion Bernstein modes we identify the eikonal function for this mode as $s|p_s|^{\frac{2N-2}{2N-1}}$. By calculating the gradient of this function we find that the refractive index n_Q of this mode is related to n_c by

$$n_Q/n_c = |p_s| = |s/a|^{\frac{1}{2N-2}} . \tag{36}$$

One can easily re-derive (36) by making use of uniform medium dispersion relation evaluated near the Nth resonance layer.

In order to evaluate the mode coupling coefficients we need to know the ratio of the components of the energy flux vectors along the direction of propagation for the fast and slow modes. The energy flux for the fast mode can readily be obtained from the Poynting theorem, giving

$$F_X = \frac{c}{4\pi} \mathrm{Re}(\mathbf{E}^* \times \mathbf{B})_x = \frac{c}{4\pi} n_c(\varepsilon_{xx}/\varepsilon_{--})^2 |E_+|^2. \tag{37}$$

The energy flux associated with the slow (Bernstein) mode is derived as in Ref.1 with ions replacing electrons, and we find

$$F_Q = \frac{c}{4\pi} \frac{N-1}{n_Q} \varepsilon_{xx} |E_+|^2. \tag{38}$$

It is clear from the above relation that for the ion Bernstein mode with $X_i >>1$ there is a 180 degree phase shift between the direction of propagation and the direction of energy flow, since ε_{xx} is negative near the resonance. We readily find that

$$\frac{F_Q}{F_X} = \frac{\mu(N-1)}{|p_s|} \left|\frac{E_+^Q}{E_+^X}\right|^2 . \tag{39}$$

We now calculate the solutions of basic scattering problems for this special case of the perpendicular propagation by taking appropriate linear combinations of the three solutions (32), (33). We readily find that when the fast

mode is incident from the high-field side the transmission coefficient for the energy flux is

$$T = e^{-\pi|b|}. \tag{40}$$

We also find that the mode converted energy flux reflection coefficient $R_M = 1 - T$, so that there is no reflected X mode in this case. The absorption within the resonance layer is zero. If the fast mode is incident from the low-field side we find that the transmission coefficient for the energy flux is again given by (40). There is a reflected fast wave in this case and $R_L = (1 - T)^2$. The transmitted mode converted coefficient for the energy flux $T_M = T(1-T)$, and the absorption is zero.

When a Bernstein mode is incident, a Bernstein mode is reflected with reflected energy flux coefficient $R_B = T^2$, where T is as defined in (40). We also find that the mode converted transmission coefficient and the mode converted reflection coefficient are the same as those just given. Again there is no absorption within the resonance layer.

As we have just shown, there is no net absorption of the incident wave energy by the plasma within the resonance layer when the relativistic corrections are dropped and the propagation is purely perpendicular. Although the energy in Bernstein waves generated by mode conversion may be eventually absorbed by the plasma, the energy deposition due to such processes does not occur in the resonance layer, and hence we do not include it here as energy flux absorption. Without the extremely small relativistic corrections we obtain energy deposition within the resonance layer only with the inclusion of a small parallel refractive index. Unfortunately, unlike the special case just studied, it is not possible to construct the solution generally in terms of quadratures. It is therefore necessary to perform a nu-

merical study for this case similar to that of Refs. 1-2. We present this study for N=3 in Sec.VI.

In order to examine (24) more efficiently, we write it as the system

$$Y' = M \cdot Y, \tag{41}$$

where $Y = (y_1, y_2, \cdots, y_{2N})$, and $y_1 = u$. The matrix M has only the following nonzero elements:

$$\begin{aligned} &M_{j,j+1} = 1, \text{ for } j=1,\cdots,N-2, \text{ or } j=N,\cdots,2N-3, \text{ or } j=2N-1, \\ &M_{N-1,N} = 1/A(s), \\ &M_{2N-2,1} = -M_{2N-2,2N-1} = 1, \\ &M_{2N,1} = \mu, \\ &M_{2N,2N-1} = -(1+\mu). \end{aligned} \tag{42}$$

It was shown in Ref. 1 that a Riccati transformation is useful not only to reduce the order of the differential equation, but also to study the asymptotic behavior of the components of Y. We define

$$z_j = y_{j+1}/y_1, \quad j=1,\cdots,N-1, \tag{43}$$

and derive the nonlinear reduced system of differential equations for z_j.

$$\begin{aligned} &z_j' = z_{j+1} - z_1 z_j, \quad j=1,\cdots,N-1, \text{ and } j=N,\cdots,2N-2, \text{ and } j=2N-2, \\ &z_{N-2}' = \frac{1}{A} z_N - z_1 z_{N-2}, \\ &z_{2N-3}' = 1 - z_{2N-2} - z_1 z_{2N-3}, \\ &z_{2N-1}' = \mu - (1+\mu) z_{2N-2} - z_1 z_{2N-1}. \end{aligned} \tag{44}$$

Since z_j' should tend to zero for $|s|$ large, we readily obtain for $|s|$ large

$$\begin{aligned} &z_j = z_1^j, \quad j=2,\cdots,N-2, \\ &z_j = A z_1^j, \quad j=N-1,\cdots,2N-3, \end{aligned}$$

$$1 - z_{2N-2} = A\, z_1^{2N-2}, \tag{45}$$

$$z_{2N-1} = z_1\, z_{2N-2},$$

$$\mu - (1+\mu) z_{2N-2} = z_1\, z_{2N-1}.$$

We see that if z_1 is of order one, then $z_{2N-2} = 1$, which in turn implies that $z_1^2 = -1$, and $z_{2N-1} = z_1$. Clearly we have obtained the X mode. Alternatively, if $z_1^{2N-2} = 1/A$, then $z_{2N-2} = \mu/z_1^2$, and $z_{2N-1} = \mu/z_1$. By selecting any one of the 2N-2 branches of $A^{-1/(2N-2)}$ we can obtain the asymptotic values of the remaining solutions

C. Reciprocity Relations

The reciprocity relations which we have obtained in the special case of purely perpendicular propagation in the nonrelativistic limit are also true generally. These relations state that (i) the transmission coefficient is the same when the fast mode is incident from the low or high field side, (ii) the reflected mode conversion coefficients are the same in the case of fast and slow mode incidences, and (iii) the transmitted mode conversion coefficients are also the same in the fast or slow mode incidences. In addition to these reciprocity relations there are two other properties which we have proved in a special case, which state that for X mode incident from the high field side there is no X mode reflected and the X mode transmission coefficient is exactly the geometrical optics result (41). In the Appendix we present a theorem which includes these results as a special case for N=2, for essentially arbitrary parallel refractive index and distribution function. The method of the Appendix could almost assuredly be applied for arbitrary N>2 to obtain the same conclusion.

Both the reciprocity theorems and the exact values of the appropriate reflection and transmission coefficients were found in the numerical calculations of second harmonic electron resonance problem in Ref.2. With slight

modifications, the theorem of the Appendix and the reciprocity proofs of this section could also be applied to electrons.

In Ref.3, for the case of electron fundamental resonance, we showed the equality of the two transmission coefficients by using the Wronskian relation. Since the order of the differential equation in that case is two, the Wronskian relation provides a bilinear relation between any two solutions of the system. In the case of the Nth resonance, the Wronskian relation is not very useful, since it relates all 2N linearly independent solutions of the system. In order to obtain a bilinear relationship between any two solutions of the system we consider the real adjoint problem. In wave propagation problems the real adjoint, as opposed to Hermitian conjugate, appears appropriate, since the complex conjugate of dispersion function, see Eqs. (28)-(25), is not an analytic function. The equation which is the real adjoint of (41) is

$$\mathbf{W}' = -\mathbf{M}^{T} \cdot \mathbf{W}, \tag{46}$$

where the superscript T denotes the transpose. If Y is a solution of the original equation, and W is a solution of the adjoint equation (46), then

$$\mathbf{Y} \cdot \mathbf{W} = \text{constant}. \tag{47}$$

After examining a few special cases of the adjoint system (46), we found that there is a simple relation for our system that connects the adjoint problem to the original problem. Specifically, we found that there is a nonsingular constant matrix P such that if Y is a solution of (41) [or W is a solution of (46)] then $\mathbf{P}^{-1} \cdot \mathbf{Y}$ satisfies (46) [or $\mathbf{P} \cdot \mathbf{W}$ satisfies (41)]. We may show directly the existence of such a matrix P. We assume that W satisfies (46), and set

$$Y = P \cdot W. \tag{48}$$

A necessary and sufficient condition for **P** to satisfy the assumed properties is that

$$M \cdot P + P \cdot M^T = 0. \tag{49}$$

We can easily construct P by direct calculation by making use of the property that it must be constant. We find that P is determined uniquely up to a multiplicative constant, and that

$$\begin{aligned} &P_{2N-k-1,k} + P_{2N-k,k-1} = 0, \text{ for } 2 \leq k \leq 2N-2, \\ &P_{2N-1,2N} = - P_{2N,2N-1} = \mu P_{2N-2,1}. \end{aligned} \tag{50}$$

To make the solution unique, we choose $P_{1,2N-2} = 1$, so that

$$\begin{aligned} &P_{2N-1-k,k} = (-1)^k, \ 1 \leq k \leq 2N-2, \\ &P_{2N-1,2N} = - P_{2N,2N-1} = - \mu. \end{aligned} \tag{51}$$

For any two solutions Y and Z of (41), the bilinear relation can then be written as

$$Y \cdot P^{-1} \cdot Z = \sum_{j=1}^{N-1} (-1)^j (Y_j, Z_{2N-1-j}) + \mu^{-1} (Y_{2N-1}, Z_{2N}) = \text{constant}, \tag{52}$$

where

$$(Y_p, Z_q) = Y_p Z_q - Y_q Z_p.$$

We now obtain the desired reciprocity relations by selecting the solutions of the simple scattering problems for Y and Z and evaluating them by use of the asymptotic expansion of Y and Z for $|s|$ large at the high-field and low-field sides. Since the asymptotic expansion of the solutions can be decomposed into incoming and outgoing fast and slow, or X and Q, components,

it is sufficient to calculate the bilinear form only for these components. We find from the expansions for $|s|$ large that the constant value of the bilinear relation (52) corresponding to the pair X and Q is zero, namely

$$X^{\rangle} \cdot P \cdot Q^{\rangle} = X^{\langle} \cdot P \cdot Q^{\rangle} = 0. \tag{53}$$

The superscripts $\rangle$ and $\langle$ denote incoming and outgoing components, respectively. For the fast mode the contribution of the sum in (52) is zero, giving

$$X^{\rangle} \cdot P \cdot X^{\langle} = -2i\mu \, |E_{+}^{X}|^2, \tag{54}$$

whereas for the slow mode only the sum gives nonzero contribution with all summands equal, so that

$$Q^{\rangle} \cdot P \cdot Q^{\langle} = -2i \, \frac{N-1}{|p_s|} \, |E_{+}^{Q}|^2. \tag{55}$$

In deriving (55) we use the branch of (31) corresponding to the propagating Bernstein mode by choosing for $|s|$ large

$$A^{\frac{k}{2N-2}} = p_s^{-k} = (\mp i)^k |p_s|^{-k}, \tag{56}$$

and we note that the first component of Q , namely Q_1, is proportional to $|p_s|^{1/2}$, cf. (34).

We now apply the bilinear form to the solution of the problems of X mode incidence from high and low field sides. These solutions for $|s|$ large can be written for the low field side incidence as:

$$\begin{aligned} U^I &= a_L^I \, X^{\rangle} + b_L^I \, X^{\langle}, \ \text{LFS}, \\ &= a_H^I \, X^{\rangle} + c^I \, Q^{\rangle}, \ \text{HFS}, \end{aligned} \tag{57}$$

and for the high field side incidence

$$
\begin{aligned}
U^{II} &= b_L^{II}\, X^{<}, \ \text{LFS}, \\
&= a_H^{II}\, X^{>} + b_H^{II}\, X^{<} + c^{II}\, Q^{>}, \ \text{HFS}.
\end{aligned}
\tag{58}
$$

Here we use the notation that the roman numeral superscripts to denote the problems I, II, III, which correspond the fast mode incident from HFS, the slow mode incidence, and the fast mode incident from the LFS, and the subscripts L and H to indicate the low and high field side components, respectively. By evaluating the bilinear form in both sides we find readily that $a_L^I b_L^{II} = a_H^I b_H^{II}$, which states that the transmission coefficients are the same for the X mode incidences from the either side, since

$$
T = T^I = T^{II} = |a_H^I / a_L^I|^2 = |b_L^{II} / b_H^{II}|^2. \tag{59}
$$

In order to obtain the reciprocity relations for the mode conversion, we consider the third basic scattering problem in which the Bernstein mode is incident. We can write this solution for $|s|$ large as

$$
\begin{aligned}
U^{III} &= b_L^{III}\, X^{<}, \ \text{LFS}, \\
&= a_H^{III}\, X^{<} + c^{III}\, Q^{>} + d^{III}\, Q^{<}, \ \text{HFS}.
\end{aligned}
\tag{60}
$$

When we apply the bilinear form to problems I and III, and evaluate the constant for both far HFS and far LFS, we obtain

$$
-2i\, a_L^I\, b_L^{III}\, \mu\, |E_+^X|^2 = -2i\, c^I\, d^{III}\, \frac{N-1}{|p_s|}\, |E_+^Q|^2. \tag{61}
$$

Multiplying both sides by their complex conjugates and rearranging the factors we easily obtain the equality of the mode coupled transmission coefficients for the fast and slow mode incidences. This establishes the second reciprocity relation.

Finally, we apply the bilinear form to the solutions of the problems II and III. In this case, the form vanishes at the far low-field side, and

evaluation at the far high-field side leads to

$$0 = 2i\, a_H^{III}\, b_H^{II}\, \mu\, |E_+^X|^2 - 2i\, c^{II}\, d^{III}\, \frac{N-1}{|p_s|}\, |E_+^Q|^2. \tag{62}$$

These reciprocity theorems are generalizations of the results of Chin and Mau' for the special case when N=2. We remind the reader that there are two more exact results previously mentioned. From the theorem in the Appendix we have shown for N=2, and we hypothesize for N>2, that (i) the transmission coefficient for the X mode incidence is given by (40), and is the same as that given by geometrical optics, and (ii) there is no reflected X mode for a high field side incident X mode.

IV. FUNDAMENTAL ION RESONANCE

We obtain field equations valid for this case by substituting (22) into Maxwell's equation, to get

$$\begin{aligned}
&(1 - X_i + \tfrac{1}{\delta} X_i a_{0,0} + \tfrac{1}{2} n_c^2 d^2) E_+ + (\tfrac{\varepsilon^2}{4\delta} X_i a_{1,0} - \tfrac{1}{2} n_c^2) d^2 E_- \\
&\qquad + i\tfrac{\varepsilon}{\delta} X_i a_{1,1} d E_{||} = 0, \qquad (63)\\
&d^2(\tfrac{\varepsilon^2}{4\delta} X_i a_{1,0} - \tfrac{1}{2} n_c^2) E_+ + (1 + \tfrac{1}{2} X_i - \tfrac{3\varepsilon^4}{64\delta} X_i d^2 a_{2,0} d^2 + \tfrac{1}{2} n_c^2 d^2) E_- \\
&\qquad - \tfrac{i\varepsilon^2}{4\delta} X_i d^2 a_{1,1} d E_{||} = 0, \\
&i\tfrac{\varepsilon}{\delta} X_i d a_{0,1} E_+ - \tfrac{i\varepsilon^2}{8\delta} X_i d a_{1,1} d^2 E_- \\
&\qquad + (1 - X_e - X_i + \tfrac{\varepsilon^2}{2\delta} X_i d a_{0,2} d + \tfrac{1}{2} n_c^2 d^2) E_{||} = 0,
\end{aligned}$$

where we have again used the charge neutrality $n_e = Z_i n_i$ to evaluate the electron contribution to the dielectric tensor. Equations (63) indicate that an appropriate scaling for our system is

$$\delta = \varepsilon^4,$$

$$E_+ = O(\varepsilon^2 E_-),\ E_{||} = O(m_e/m_i E_-).$$

By dropping the terms of order m_e/m_i, and eliminating E_-, we obtain

$$[1 + d^2 + d^2 A(s) d^2] E_- = 0, \tag{64}$$

where d and s are as given in (24) and (25) with N=1, and

$$A(s) = \frac{n_c^2}{8X_i} \frac{4a_{1,0}^2 - 3a_{2,0}a_{0,0}}{4a_{0,0}},$$
$$n_c^2 = 2 + X_i - n_{||}^2 .$$

We identify n_c as the cold plasma refractive index for the magnetosonic (fast) mode evaluated at the resonance. We may generally ignore the last term in the definition of n_c in (64), since, as we show by numerical computation in Sec.VI, geometrical optics becomes valid whenever $n_{||}$ is large enough to make significant to make a significant modification to n_c.

We may extract some conclusions from a simple examination of (64). First, although we have carried relativistic modifications of the Vlasov equation, it should be no surprise that relativistic effects in the ions are negligible. Specifically, the relativistic correction included in the factor Δ_1 defined in (4) is small when

$$n_c^2 \approx X_i >> \eta^{-2}.$$

We restrict our study to this range, so that $a_{n,0}$ functions may be replaced by the standard plasma dispersion functions. Second, the Doppler broadening effects generated by the term containing the parallel refractive index dominate spatial gradients in the boundary layer when

$$n_{||} >> \eta^3 X_i^2.$$

When the latter condition is satisfied we expect the geometrical optics approximation to hold within the resonance layer. We verify this result in our numerical investigation which we present in Sec.VI. Third, since the order of Eq.(64) is four we expect mode coupling to some other mode to oc-

cur. We identify this mode in subsection B by examining the uniform medium dispersion relation near the resonance layer. Finally, we note that (64) can be solved by quadratures for the case of purely perpendicular propagation in the nonrelativistic limit. This special case is of particular importance since it also corresponds to the large |s| limit of Eq.(64). We present this limit in the following subsection.

A. Asymptotic Solutions

For |s| large, we replace A(s) in (64) by

$$A(s) = a/s,$$

where

$$a = (\tfrac{3}{4}\langle w_\perp^4\rangle - \langle w_\perp^2\rangle^2) n_c^3 \delta(L\omega/c)/(8X_i). \tag{65}$$

For a Maxwellian equilibrium distribution function the first factor in the right-hand side of (65) is 1/2.

Four linearly independent solutions of

$$[1 + d^2 + d^2\,(a/s)\,d^2]u = 0$$

are

$$u = a \int_{C_i} dp\, \frac{p^2}{p^2+1} \exp[(s-a)p + \frac{ap^3}{3} + \frac{a}{2i} \log\frac{p-i}{p+i}], \tag{66}$$

where the contours C_i, i=1,2,3,4, are sketched in Fig.1. Contributions to these solutions for |s| large come (i) from the branch points $p_b \mp i$, and (ii) from the stationary points $p_s^2 = (a-s)/a$. The first contribution generates the fast branch of the extraordinary mode, whereas the latter corresponds to the slow branch which we study in the geometrical optics limit the next subsection. We omit the solution corresponding to C_4, which grows exponentially in the low field side and thus is of no interest in the solutions of simple scattering problems. For the remaining three solutions we obtain for |s| large:

$$u_0 = e^{3a\pi/4}\,(2\sinh\tfrac{a\pi}{2})(p_+ + p_-) + (q_+ + q_-),\ \text{HFS}$$
$$= 0,\ \text{LFS}, \qquad (67)$$
$$u_\mp = p_\mp\, e^{5a\pi/4} + q_\mp,\ \text{HFS}$$
$$= p_\mp\, e^{3a\pi/4},\ \text{LFS},$$

where

$$p_\mp = -\frac{a}{2i}\,\Gamma(\mp\tfrac{a}{2i})\exp[\mp i(s-\tfrac{4}{3}a+\tfrac{a}{2}\log 2|s-2a|)],$$

$$q_\mp = \left|\frac{a\pi}{p_s}\right|^{1/2}\frac{p_s^2 e^{a\pi}}{p_s^2+1}\exp[\mp i\tfrac{\pi}{4}\mp i\tfrac{2}{3}|p_s|(s-a)].$$

The basic scattering problems in this limiting case can be constructed as appropriate linear combinations of these solutions. First, we must investigate the geometrical optics region near the resonance layer, and match our fast and slow branches represented by p and q functions given above to those predicted by the uniform field dispersion relation. We carry out this analysis in the next section in which we also determine the energy flux associated the various waves.

B. Geometrical optics region

Consistent with the formulation given in subsection A, we assume that the geometrical optics assumptions hold for the system at least outside of the resonance layer. We start with the linearized Vlasov equation with plane waves propagating perpendicular to the equilibrium magnetic field, and calculate the dielectric tensor in the usual manner. We employ the usual (+,-) representation, and evaluate the tensor elements near the fundamental resonance to obtain

$$e_{++} = 1 - X_i + \tfrac{1}{2}X_i[G_{1,1} + \tfrac{1}{2}G_{2,2} - G_{0,0}/(1-\omega/\Omega)],$$

$$\varepsilon_{--} = 1 + X_i + \tfrac{1}{2}X_i[G_{1,1} + \tfrac{1}{2}G_{0,0} - G_{2,2}/(1-\omega/\Omega)],$$

$$\varepsilon_{+-} = \varepsilon_{-+} = \tfrac{1}{2}X_i[G_{1,1} - G_{2,0}/(1-\omega/\Omega)],$$

where

$$G_{i,j} = \int d^3w\, w_\perp \frac{\partial F}{\partial w_\perp} J_i J_j,$$

J_k are the Bessel functions with arguments $n\eta w_\perp \omega/\Omega$, and n is the local perpendicular index of refraction. We obtain the dispersion relation valid near the resonance as

$$n^2 = 2 + X_i + \frac{1}{1-\omega/\Omega} X_i \frac{G_{0,2}^2 - G_{0,0}G_{2,2}}{G_{0,0}} .$$

If we keep only the lowest significant order terms in η, we find a biquadratic dispersion relation:

$$n^2 = 2 + X_i + \frac{\eta^4}{1-\omega/\Omega} X_i \frac{1}{8} n^4 (\tfrac{3}{4}\langle w_\perp^4\rangle - \langle w_\perp^2\rangle). \qquad (68)$$

We readily observe that if we replace d^2 in (66) by $-(n/n_c)^2$ we recover (68). For $\eta^4/(1-\omega/\Omega)$ small, one root of (68) is approximately n_f, corresponding to a fast wave, where

$$n_f^2 = n_c^2 = 2 + X_i \approx X_i. \qquad (69)$$

The other root n_s, again for $\eta^4/(1-\omega/\Omega)$ small, corresponds to the slow wave, $n_s^2 >> n_f^2$, where

$$n_s^2 \approx 16X_i^2 \frac{1-\omega/\Omega}{e^4}$$

When we rewrite (68) in the more convenient form

$$n^2 = n_f^2 + \frac{a}{s} \frac{n^4}{n_f^2} ,$$

we easily see that the slow mode propagates only on the high field side, and coalesces with the fast mode at s=4a, where $n^2=2n_f^2$. No modes propagate from

this value of s to resonance.[8] We can relate the refractive indices of the slow and fast modes by

$$n_s^2 = \frac{s}{a}\, n_f^2 \approx -p_s^2\, n_f^2. \tag{70}$$

Figure 2 illustrates these modes in the $(n,\omega/\Omega)$ plane for a fixed value of X_i and T_i. We may expect that some portion of the fast wave energy incident from the LFS should tunnel through the resonance layer to the HFS and partially mode coupled to the slow mode. Similarly, HFS incident fast or slow modes may suffer reflection and mode conversion in addition to tunnelling to the LFS as a fast mode. We next establish the relation between the energy flux associated with these modes, which is needed for the scattering problems.

The component of the energy flux along the direction of propagation, F_x, for our problem is

$$F_x = \frac{c}{4\pi}\, n_c\, \mathrm{Im}\{E_y^* dE_y\}.$$

Since $E_+ = O(\varepsilon^2 E_-)$, we may replace E_y by $\frac{i}{2} E_-$, and

$$\frac{F_x^s}{F_x^f} = \frac{n_s}{n_f} \frac{|E_-^s|^2}{|E_-^f|^2} = |p_s| \frac{|E_-^s|^2}{|E_-^f|^2}. \tag{71}$$

We can now calculate the solutions of the basic wave propagation problems when $n_{||} = 0$.

By making use of asymptotic expansions (67) of the solutions to the wave equation (66) we can readily calculate the solutions of the basic scattering problems. When the fast mode is incident from the HFS the solution of the problem may be represented by u_-. We find that the transmission coefficient (or the fraction of the transmitted energy flux) for the fast mode incidence is

$$T_X = T = e^{-a\pi}, \tag{72}$$

where a is given in (65). In this case, there is no reflected fast mode, and the energy not transmitted is fully mode converted to the slow mode. i.e. $R_M = 1-T$. To obtain these results, we use the energy flux ratios evaluated in (71).

When the fast mode is incident from the LFS the solution is that linear combination of u_0 and u_- which eliminates p_+ on the HFS, and we find that the transmission coefficient is again $T_X = T$, the reflection coefficient R_L equals $(1-T)^2$, and the mode conversion coefficient is $T_M = T(1-T)$. For the slow mode incidence the reflection coefficient is $R_S = T^2$, and the transmitted and reflected mode conversion coefficients are the same as the case of fast mode incidence. For any incident wave there is no net energy absorption for $n_{||} = 0$. For non-zero energy absorption we must assume $n_{||} \neq 0$, and we rely on numerical computations of Sec.VI.

C. Reciprocity relations

We write (64) in the form

$$u' = M \cdot u, \tag{73}$$

where the prime denotes derivative with respect to s, the components of the vector u are $u_1 = E_-$, $u_2 = u_1'$, $u_3 = A(s)u_2' + u_1$, $u_4 = u_3'$, and

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/A & 0 & 1/A & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

One can easily show that if u satisfies (73), then $v = P \cdot u$ satisfies the real adjoint equation, namely

$$v' = -M^T \cdot v, \tag{74}$$

where

$$P = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

We hence conclude that if u and $\bar{u}$ are any two solutions of (73) then the bilinear form

$$u \cdot P \cdot \bar{u} = \text{constant}.$$

The reciprocity relations are then readily obtained by direct evaluation of the constant at the far LFS and far HFS for any pair of solutions of two distinct basic scattering problems. We suppress the details of these calculation here. We find that the transmission coefficient for the fast mode incidence is independent of the direction of incidence, and the reflected (or transmitted) mode coupling coefficients are the same in the case of fast mode converted into slow mode or conversely. In proving these relations we break the solutions under consideration into incoming and outgoing fast and slow components, and utilize their asymptotic forms for $|s|$ large, which we derive in the next section.

The theorem of the Appendix applied in this case shows that the transmission coefficient for incident fast mode is given exactly by (65), which again matches the geometrical optics result. Also, when the fast mode is incident from the high field side there is no reflected fast mode.

D. Reformulation of the general case

The existence of solutions growing exponentially for $|s|$ large in the LFS makes it necessary to use some special numerical integration techniques. We follow closely the methods described in detail for the problem of second harmonic electron cyclotron resonance.[1] We first transform (73) using a Riccati reduction into a nonlinear third order system by setting

$$w_i = u_{i+1}/u_1, \quad i=1,2,3. \tag{75}$$

This transformation essentially reduces the exponential growth of the solution into algebraic growth, and it avoids the occurrence of very large numbers. We find

$$\begin{aligned} w_1' &= (w_2 - 1)/A(s) - w_1^2, \\ w_2' &= w_3 - w_1 w_2, \\ w_3' &= -1 - w_3 w_1. \end{aligned} \tag{76}$$

We integrate (76) from far LFS ($s \to -\infty$) into the resonance region ($s=0$) to obtain one solution which vanishes asymptotically in far LFS. In order to obtain two other solutions which behave like incoming and outgoing fast mode in the LFS we use the method of reduction of order by which the growing solutions of (74) are suppressed. If we denote by y the solution which is exponentially small for $s \to -\infty$, the other two solutions u are given by solutions z of the reduced system by the relations

$$u = z + \lambda y, \tag{77}$$

$$z' = M \cdot z - (z_2/y_1)y, \tag{78}$$

where we have set $z_1 = 0$, so that $\lambda = u_1/y_1$, and $\lambda' = z_2/y_1$.

For $s>0$ all solutions of (74) represent propagating waves, so that we may return to (74). We extend each of the three solutions for $s<0$ into the domain $s>0$, and for convenience we choose $\lambda = 0$ and $y_1 = 1$ at $s=0$. We integrate these solutions for s large and positive (far HFS), and we decompose numerically the solutions so obtained into incoming and outgoing, fast and slow waves. The solutions of basic scattering problems can then be obtained easily by taking appropriate linear combinations of these solutions. Finally, by making use of the energy flux theorems, we calculate the propagation characteristics for each problem. Numerical results are presented in

Sec.VI.

V. ION-ION HYBRID RESONANCE

We consider a plasma containing two species of ions for which one species is at fundamental resonance while the other is at second harmonic resonance. To be more explicit we refer to these species as F and S which may represent a H-D or a ^{3}He-T system. Although the equations for the species F are similar to those given in the previous section, there are several important differences which introduce confusion. We define the boundary layer variable as

$$\xi = \frac{1}{\delta}\left[\frac{1}{2} - \frac{\Omega_S(x/L)}{\omega}\right], \tag{79}$$

or equivalently

$$= \frac{1}{2\delta}\left[1 - \frac{\Omega_F(x/L)}{\omega}\right].$$

Note that (79) is to be contrasted with (2). In the current formulas (18) and (22), (2) is used to define ξ, while in this section (79) defines ξ differently for each species at resonance. Thus, we must be careful in the evaluation of the ion currents. We introduce the parameters η_F and η_S, where

$$\eta_{F,S} = (v_{th})_{F,S}/c. \tag{80}$$

which we assume to be on the same order of magnitude, and we assume

$$\varepsilon_{F,S} = \eta_{F,S}\, n_c \ll 1,$$

where n_c is the appropriate cold plasma refractive index, which, together with the stretching factor δ, we specify later. We find the ion current to be the sum of (18) and (22). We readily observe that our boundary layer

expansion is consistent provided that

$$\delta = \varepsilon_S^2,$$
$$X_F/X_S = (Z_F/Z_S)^2(n_F/n_S)(m_F/m_S) = 2(Z_S/Z_F)(n_F/n_S) = O(\delta). \tag{81}$$

We add the electron current obtained from the charge neutrality, drop terms of order m_e/m_i, and eliminate E_-, to obtain

$$\{1 + d^2 + (1 + \mu + d^2)[dA(s)d - B(s)]\}E_+ = 0, \tag{82}$$

where

$$\mu = (\varepsilon_{--} - n_{||}^2)/(\varepsilon_{++} - n_{||}^2),$$
$$\varepsilon_{++} = 1 - 4\,X_S,$$
$$\varepsilon_{--} = 1 + \frac{4}{3}\,X_S,$$
$$\varepsilon_{xx} = (\varepsilon_{++} + \varepsilon_{--})/2,$$
$$n_c^2 = (\varepsilon_{--} - n_{||}^2)(\varepsilon_{++} - n_{||}^2)/(\varepsilon_{xx} - n_{||}^2),$$
$$d = d/ds,$$
$$s = -2\xi\delta n_c L\omega/c,$$
$$A(s) = X_S\, a_{1,0}^S/(\varepsilon_{xx} - n_{||}^2),$$
$$B(s) = \frac{X_F\, a_{0,0}^F}{\varepsilon_{xx} - n_{||}^2}.$$

According to (81) the density ratio $n_F/n_S = O(\delta)$ and this might seem to be an uninteresting parameter range. However, the numerical calculations given in Sec.VI show that a maximum in the wave energy absorption occurs with a rather small addition of minority species on the order of 1% of the electron number density. Such a small minority fraction is consistent with our ordering. We note that if the ordering (81) is not satisfied, within the layer E_+ must get much smaller in magnitude in order to keep the first term in the right-hand side of the first equation in (22) balanced. We find

that in this case a second boundary layer exits with $\delta = \varepsilon^4$ within which (64) is satisfied, with $n_C^2 = 2 - X_F + 8X_S/3 - 2n_{||}^2$. For all practical purposes, however, the inner boundary layer is almost transparent, since the transmission coefficient determined by (72) is very nearly one.

We have not found explicit integral representations of solutions of the asymptotic form of (82) for $|s|$ large. We may however extract necessary informations about the asymptotic behavior of solutions of (82) by making use of standard techniques. We write (82) in the form

$$u' = M \cdot u, \tag{83}$$

where the prime denotes derivative with respect to s, the components of the vector u are $u_1 = E_+$, $u_2 = A(s)u_1'$, $u_3 = u_1 + u_2' - B(s)u_1$, $u_4 = u_3'$, and

$$M = \begin{bmatrix} 0 & 1/A & 0 & 0 \\ B-1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu & 0 & -1-\mu & 0 \end{bmatrix}. \tag{84}$$

If we set $w_j = u_{j+1}/u_1$, $j=1,2,3$, we find the Riccati reduced system for our problem may be written as

$$\begin{aligned} w_1' &= w_2 - 1 + B(s) - w_1^2/A(s), \\ w_2' &= w_3 - w_1 w_2/A(s), \\ w_3' &= \mu - (1+\mu)w_2 - w_1 w_3/A(s). \end{aligned} \tag{85}$$

We construct asymptotic expansions of solutions of (85) in terms of asymptotic series in reciprocal powers of s. For the X mode, we find

$$\begin{aligned} w_1 &= \mp iA\,[1 - \tfrac{\mu}{2}(A+B)], \\ w_2 &= 1 - A - B, \\ w_3 &= \mp i\,[1 - (1 - \tfrac{\mu}{2})(A+B)], \end{aligned} \tag{86}$$

where A and B denote a/s and b/s, respectively, and

$$a = -\frac{X_S\, \delta n_c L\omega/c}{\varepsilon_{xx}-n_{\|}^2},$$

$$b = -\frac{X_F\, n_c L\omega/c}{2(\varepsilon_{xx}-n_{\|}^2)}. \tag{87}$$

For the Bernstein mode

$$\begin{aligned} w_1 &= \mp iA^{1/2}(1 + \tfrac{\mu}{2}A - \tfrac{1}{2}B) + \frac{A}{4s}, \\ w_2 &= -\mu A(1 + A + B) \pm i\mu A^{3/2}/s, \\ w_3 &= \pm i\mu A^{1/2}[1 + (1 + \tfrac{\mu}{2})A + \tfrac{1}{2}B] - \mu\frac{A}{4s}. \end{aligned} \tag{88}$$

Since $u_1'/u_1 \equiv w_1/A$ is proportional to the derivative of the eikonal function for $|s|$ large, we readily find for the X mode

$$u_1 = e^{\mp i[s - (\mu/2)(a+b)\log s]}. \tag{89}$$

For the Bernstein mode, we obtain

$$u_1 = s^{1/4}\, e^{\mp iA^{-1/2}(\frac{2}{3}s+\mu a-b)}. \tag{90}$$

The theorem of the Appendix applied to this case shows that the transmission coefficient for the X mode is

$$T_X = e^{-|\mu(a+b)|\pi}. \tag{91}$$

Furthermore the reflection coefficient is zero for the X mode in high field side incidence. We next match these results to the geometrical optics results evaluated near the resonance layer, and establish the energy flux relations.

A. Geometrical optics results

For the present problem, we can approximate the dispersion relation for the Bernstein mode as $\varepsilon_{xx} = 0$, which, when evaluated near the resonance, yields the relation between the refractive index of the qnasi-electrostatic (Bernstein) mode and n_c:

$$\frac{n_Q^2}{n_c^2} = \frac{1}{A} + \frac{X_F}{\delta X_S} \approx \frac{s}{a}. \tag{92}$$

Eqnation (92) also is a direct consequence of (90).

When we evaluate the component of the energy flux along the direction of propagation for the Bernstein mode we find that the contribution from the species F is negligible near the resonance layer, so that

$$\frac{F_x^Q}{F_x^X} = \mu |A|^{1/2} \frac{|E_+^Q|^2}{|E_+^X|^2} . \tag{93}$$

We use (93) to calculate the mode coupling coefficients associated with the scattering problems.

Before we present numerical results, we give some analytical results directly obtained from (82). First we prove that the reciprocity relations are still true for the ion-ion hybrid case discussed here.

B. Reciprocity relations for the ion-ion hybrid resonance

The proof of the reciprocity relations can be established in a manner analogous to the previous case. We find that if

$$P = \begin{bmatrix} 0 & \mu & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

then if u satisfies (83) then v=P·u satisfies the real adjoint equation

$$v' = -M^T \cdot v.$$

Hence, if u and $\bar{u}$ any two solutions of (83), then the bilinear form

$$u \cdot P \cdot \bar{u} = \text{constant}.$$

By evaluating the constant at far LFS and far HFS for the solutions of the basic scattering problems, we again derive the same reciprocity relations

as for the other resonances.

VI. NUMERICAL RESULTS

A. Absorption at the third cyclotron harmonic resonance

The results obtained in Sec.III indicate that, because b in the expression (40) is typically large, when the propagation is purely perpendicular to the equilibrium magnetic fieldfast waves incident the Nth ion cyclotron harmonic resonance layer are mainly reflected for the low field side incidence, and converted to reflected ion Bernstein waves for the high field side incidence. When the propagation is not purely perpendicular exact solutions are not available, and we turn to numerical computation. In this section we study wave propagation near third harmonic resonance. We find that for the low field incident fast mode, when $n_{||}$ is approximately 5 or larger, reflection and mode conversion of this mode become small, so that all incoming wave energy which is not transmitted to the high field side is absorbed by the plasma. The rate of decrease in the reflection and mode conversion coefficients as a function of $n_{||}$ increases, and the corresponding increase in the absorption coefficient, is largely independent of the plasma temperature or density. However, this rate increases with the characteristic length. We thus conclude that these waves may provide an efficient way of heating a plasma, particularly in a large, moderately hot tokamak.

For waves incident from the high field side, as a consequence of the much slower rates of decrease of the reflected mode conversion coefficient with increasing parallel refractive index, the absorption rates are much smaller. When $n_{||}$ becomes sufficiently large, the absorption is further reduced, since the transmission coefficient becomes larger due to geometrical effects. The absorption coefficients for the high and low field incident fast mode eventually equal one minus transmission coefficient, and geo-

metrical optics becomes valid. The value of $n_{||}$ in this region is quite high so large that accessibility of waves may be poor. For the case of Bernstein mode incidence we find that the absorption is poor when $n_{||}$ is small, which is mainly caused by the large reflected mode conversion, and the absorption sharply approaches one when $n_{||}$ is sufficiently large.

In our numerical study, we utilize a two-fold reduction of order technique to eliminate the two linearly independent solutions which grow on the low field side. By integrating towards the resonance we calculate numerically two solutions which are zero asymptotically . In addition we also calculate two (incoming and outgoing) fast mode components. On the high field side there is only one solution which grows, and we eliminate it by the same method given in our treatment of second harmonic resonance.[2,3] By integrating again towards the resonance, we calculate one solution which is asymptotically zero and four solutions which are propagating fast and slow modes. We match the solutions computed on the low field side to the solutions computed on the high field side at the resonance, and thus obtain solutions valid within the entire boundary layer, giving the solutions of the basic scattering problems. We also use the reciprocity relations obtained in Sec.III to improve the numerical accuracy of the results which we present next.

In Fig.3 we present the wave propagation characteristics in a hydrogen plasma for various values of $X = (\omega_p/\omega)^2$ as a function of $n_{||}$, for T = 10 keV, and $L\omega/c = 1$, where L is the characteristic length. The results just stated are clearly evident. We see that as $n_{||}$ increases, the transmitted mode conversion coefficient, T_M, and reflection coefficient for the low field incident fast mode, R_L, sharply approach zero. A_L, the absorption coefficient for the low field incident fast mode, then equals $1 - T_X$, as expected by geometrical optics. The rate of decline in the reflected mode

coupling coefficient, R_M, as a function of $n_{||}$ is much slower than T_M and R_L, and it depends on the value of X. As a result of this the absorption coefficient for the high field incident fast mode is more sharply peaked at a much lower value that that of the low field incidence. The absorption coefficient A_B for the Bernstein mode incidence increases monotonically with $n_{||}$. We find similar behavior for other values of T.

Figure 4 depicts the absorption contours for the low and high field incident fast mode in the $(X,n_{||})$ plane. We also present similar results in the $(T,n_{||})$ plane in Fig.5. We study the effect of the machine size on absorption in Fig.6. We see that as $L\omega/c$ increases the mode coupling approaches zero more rapidly with $n_{||}$, which results in higher absorption rates.

B. Wave propagation at the fundamental ion resonance

In Sec.IV we have shown that the boundary layer method can be applied successfully to case of fundamental resonance, where within a narrow resonance region the fast and slow modes bifurcate and geometrical optics fails. We obtained an analytical expression for the transmission coefficient for the case of fast mode incidence. Reflection and mode conversion coefficients are also given for the case of purely perpendicular propagation. In the present subsection we numerically study the case when $n_{||}$ is not zero for an hydrogen plasma. We show that reasonable absorption rates of the incident fast mode energy is possible if the ion temperature is greater than 10 Kev, and density is sufficiently high (X>1000), when $n_{||} > 2$.

Since the field equation is similar to that studied for the second electron resonance which we have studied in Refs.1-2, we employ the same numerical method in solving (73). We find that for $T \leq 5$ keV the incident fast magnetosonic mode largely tunnels through the resonance layer, causing

little absorption, even for densities so high that X = 10000. For T = 15 keV respectable absorption rates may be achieved at larger values of X and moderate values of $n_{||}$. We present these results in Fig.7 where the absorption contours in the $(X, n_{||})$ are plotted for T = 10 kev. Figure 8 depicts a similar plot for T = 15 keV. Geometrical optics is valid when the absorption rates for the HFS and LFS incidences approximately equal each other, namely $A_H \approx A_L$.

We find quite respectable absorption rates for the incident slow mode even when both density and parallel refractive index are small. We thus conclude that the latter may be considered as a potential heating scheme for tokamak plasmas if these waves are accessible at the resonance layer.

C. Numerical study for the ion-ion hybrid resonance

We have shown in Sec.V that when two ion species are present in the plasma, one at the fundamental, and the other at the second harmonic resonance at the same resonance surface, the energy transmission coefficient given by (92) decreases. Our formulation is also valid in the limit when the minority ion density approaches zero, so that the numerical results presented in this section contains as a special case the results with only one ion species resonant at the second harmonic. Although we have analytic results for the latter case when $n_{||}$ is zero, we still have to carry out a numerical study for the two-ion problem.

When the minority density becomes sufficiently large, the resonant term in J_+ is balanced by E_+, whose magnitude becomes small as waves get close to the resonance, and waves transfer into the magnetosonic mode. As we have indicated in Sec.V, a second boundary layer is formed in this case which is much closer to the exact resonance surface. The result of the previous subsection shows that incident fast waves very nearly all tunnel through the

resonance, unless X_F is extremely large. We thus may ignore the effect of the inner layer when tokamak-like plasmas are being studied. For every tokamak configuration that we have studied the maximum increase in the energy absorption rate caused by the minority species occurs when the minority ion density is somewhat less than one percent of the electron density. The Bernstein mode is not affected by the inner boundary layer. Hence, the effect of large concentrations of fundamental species need not be considered.

We present our results in Figs.9-11, where the absorption contours are plotted in the plane of minority density and the parallel wavenumber. We have selected three examples corresponding to typical PLT, JET, and CIT parameter ranges. Figure 9 depicts the case of a (H-D) plasma with $B_0 = 3$ T, $n_e = 3 \times 10^{13}$ cm^{-3}, $T_H = T_D = 2$ keV and $L = 120$ cm. We see that although the fast mode incidence yields rather poor absorption rates, the incident Bernstein mode can be used to heat the plasma quite efficiently. In Fig.10, we give the results for a (H-D) plasma with $B_0 = 3.45$ T, $n_e = 3.5 \times 10^{13}$ cm^{-3}, $T_H = T_D = 5$ keV and $L = 300$ cm. We find good absorption rates for the low field side incident fast mode even for quite small values of parallel wavenumber when n_H is about $.01 n_e$. The absorption of the incident Bernstein mode is appreciable at moderate values of $k_{||}$. Finally, in Fig.11 we present the absorption level lines for a (^{3}He-T) plasma with $B_0 = 10$ T, $n_e = 5 \times 10^{14}$ cm^{-3}, $T_{He} = T_T = 10$ keV and $L = 122$ cm. We find that if the waves are incident from the high field side they suffer a nearly full reflected mode conversion, since $R_M \sim 1$. We find excellent absorption rates for fast waves incident from the low field side when the minority density is 0.5% of the electron density. More energy is deposited to the majority ions at larger values of parallel wavenumber. We also give the deposition profiles for the ion species within the resonance layer in Fig.12 for a typical case

of $k_{||} = 2\ m^{-1}$ and $n_{He} = 0.005\ n_e$. We see that the energy deposition is quite localized for this case.

The higher absorption rates that can be obtained for larger values of the parallel wavenumber are due to Doppler effects, rather than resonance effects, hence this domain should be avoided when accessibility is a problem. On the other hand, if $k_{||}$ is too small, we find that most of the energy is deposited to the fundamental species.

VII. DISCUSSION AND CONCLUSIONS

This paper contains analytical and computational results dealing with wave propagation across the Nth ion cyclotron resonance layer where the geometrical optics approximation is not valid. We have used a boundary layer expansion, and when it is valid we can solve the linearized Vlasov equation order by order within the resonance layer. We find that the Nth Fourier component ($N \geq 2$) in gyrophase angle of the perturbed distribution function becomes large in magnitude within the resonance layer, so that there is a zeroth order contribution to the current density from the terms involving the spatial derivatives. When N=1, the current contribution is known to be large. An explicit evaluation of the resonance layer current gives rise to field equations valid within the layer for the cases of pure Nth harmonic resonance, $N \geq 1$, and for the case of ion-ion hybrid resonance. In most cases with $n_{||} = 0$ we may represent the solutions by contour integrals, in all cases we may determine the transmission coefficient for the incident fast wave exactly, cf. Eqs.(40), (72), and (92).

For pure Nth harmonic resonance, $N \geq 2$, and for a given value of the effective expansion parameter $\varepsilon K(0)$, the optical thickness (or the transmission coefficient T) takes a minimum value (correspondingly a maximum for T) when X_i satisfies the cubic equation

$$N^6X^3 + 2N^4(3-3N-N^2-N^3)X^2 + (8N-9)N^2(N^2-1)X - 2(N-2) = 0.$$

For N = 2 this equation has three real roots, 0, 0.18, 7.32, and for larger values of N the real root is approximately $2(N + 1 + 3/N - 3/N^2)$. From this result we easily infer that when N is greater than 2, for almost every case of physical interest (with the possible exception of some astrophysical applications where L may be very large), T is nearly zero. For example, for N = 3 we find $T \sim \exp(-429.4/|K(0)|) = \exp(-429.4\varepsilon^4 k_c L)$, where $k_c = N\omega_{pi}/c$ is the cold plasma wavenumber for the X mode. Therefore, the fast mode incident from the high field side is almost fully mode converted, whereas there is an almost full reflection of this mode when the incidence is from the low field side. This result clearly indicates the breakdown of the geometrical optics expansion near the resonance layer.

In Secs.III-V and the Appendix we have presented analytical proofs of various properties of the scattering problems. We have shown that reciprocity relations hold within the limits of our expansions. These relations state that the transmission coefficient is independent of the direction of the incidence even when the relativistic corrections and the effects of the nonzero $n_{||}$ are included. Further, the mode converted transmission coefficients for the fast and slow mode incidences are equal. Similarly, the mode converted reflection reflection coefficients for the fast and slow mode incidence are equal. These properties are very useful in simplifying numerical matching, and they can also be used to check the accuracy of the numerical results. We have also proven a more difficult proposition, which gives explicit values for the transmission coefficient for the incident fast waves, and shows that there is no reflection for the fast mode incidence from the high field side. These proofs are given only for the special cases of fundamental, second harmonic, and ion-ion hybrid resonance. For higher ion harmonics the proof becomes too complicated to be considered here although

we believe that the proposition is still true.

Section VI contains the results of numerical studies. When $n_{||}$ is sufficiently large we recover geometrical optics since the coefficients for wave reflection and moda conversion to Bernstein mode become small. In this limit the energy absorption becomes approximately one minus transmission. We first considered the special case of N=3. On the basis of this study we conclude that waves propagating with frequencies equal to the third ion cyclotron harmonic provide an efficient means for heating the plasma, particularly so for the case of the low field incident fast mode with a small $n_{||}$. We have also shown that even smaller values of $n_{||}$ are needed for larger tokamaks. Hence one may achieve better accessibility of incident waves in this case. The absorption of the HFS incident fast mode and/or the Bernstein mode incidence require considerably higher values of $n_{||}$ in order to achieve reasonable absorption, thus they are not as promising as the LFS incidence. We also find that the absorption is more efficient for larger ion temperatures.

The case of magnetosonic waves incident on the fundamental resonance layer when $n_{||}$ is not zero is also studied numerically. We have found that reasonable absorption rates of the incident fast mode energy is possible if the ion temperature is greater than 10 Kev, and density is sufficiently high ($X \gg 1000$), when $n_{||} > 2$. We find, however, that in the case of the slow mode incidence reasonable absorption rates may be achieved for the parameter range comparable to tokamaks which exists today.

In the case of two ion hybrid resonance, we find that very promising improvement in absorption is obtained, particularly when the minority density is approximately one percent of the electron density. At this value there is a strong peak in the absorption rate for LFS incidence when the

parallel refractive index is sufficiently small. We have also calculated energy deposition profiles for each species. When the plasma parameters are chosen appropriately, we find that the plasma heating by this method is quite favorable.

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APPENDIX

In this appendix we prove a property of asymptotic expansions of solutions of differential equations appropriate to the cases of fundamental, hybrid, and second harmonic resonance. From this result it is possible to obtain an explicit formula for the transmission coefficient for an incident fast wave, and to show that for a fast wave incident from the high field side no fast wave is reflected. An extension of the proof could no doubt be given for any higher harmonic resonance, but we do not explore this possibility.

We consider a system of four, linear, first order differential equations, which we write in the form

$$y' = A(s)\, y, \tag{A1}$$

where $A(s)$ is a four by four matrix and $y(s)$ is a four vector. We assume that $A(s)$ is analytic in $\mathrm{Im}\ s < 0$, and continuous in $\mathrm{Im}\ s \leq 0$. All diagonal elements of $A(s)$ vanish, and more generally $\mathrm{Tr}\ A(s) = 0$. Additionally, $A(s)$ possesses an asymptotic expansion for $|s|$ large valid in $\mathrm{Im}\ s \leq 0$

$$A(s) = s \sum_{n=0}^{\infty} \alpha_n / s^n, \tag{A2}$$

so that for any integer M there is a constant K_M such that

$$|A(s) - s\sum_{n=0}^{M} \alpha_n/s^n| \leq K_M|s|^{-M} \tag{A3}$$

All the matrices α_n are real and the diagonal elements of all α_n vanish. All solutions of (A1) are analytic in Im s < 0 and continuous in Im s $\leq$ 0. In addition, for any four vector solutions of (A1), say , z_1, z_2, z_3, and z_4 the matrix $M = (z_1\ z_2\ z_3\ z_4)$ has a constant determinant.

For equations of the form (A1), where asymptotic expansions of A(s) of the form (A2) are assumed, it is possible to obtain formal asymptotic expansions of solutions which are of the form of the product of a power of s, the exponential of a polynomial in a fractional power of s, and a reciprocal power series in a fractional power of s. Such asymptotic expansions are developed in Coddington and Levinson.[20] It is also shown there that on any ray, arg(s) fixed, there are solutions of the original system which are asymptotic to these series which have been formally obtained. The asymptotic series are uniquely determined by the coefficient matrices, and conversely, if the formal series are given to some order, then the coefficient matrices are also known to some order.

Appropriate to the systems of differential equations of interest to us, we assume that there are four formal asymptotic series of the form

$$y_1 = \exp\{is + i(b/2)\log s\}\sum_{n=0}^{\infty} \beta_n/s^n, \tag{A4}$$

$$y_2 = \exp\{-is - i(b/2)\log s\}\sum_{n=0}^{\infty} \beta_n^*/s^n, \tag{A5}$$

$$y_3 = s^{1/4}\exp(cs^{3/2} + d\,s^{1/2})\sum_{n=0}^{\infty} \gamma_n/s^{(n/2)}, \tag{A6}$$

$$y_4 = s^{1/4}\exp(-cs^{3/2} - d\,s^{1/2})\sum_{n=0}^{\infty} \delta_n/s^{(n/2)}, \tag{A7}$$

where b, c, d, are real, and b and c are positive. In addition β_0, γ_0, and

δ_0 are nonzero. For any formal series of the form (A2) or (A4)-(A7) we introduce the notation of superscript P to denote the series truncated to terms of order s^{-P} times any possible exponential factor, thus

$$A^P = \sum_{n=0}^{P\pm1} s \alpha_n / s^n,$$

$$y_1^P = \exp\{is + i(b/2)\log s\} \sum_{n=0}^{P} \beta_n / s^n,$$

$$y_3^P = s^{1/4} \exp(cs^{3/2} + d\, s^{1/2}) \sum_{n=0}^{2P} \gamma_n / s^{(n/2)},$$

and so on. On any ray arg(s) fixed there are four solutions of (A1) which are asymptotic to y_1^P, y_2^P, y_3^P, y_4^P. For the exact solutions it follows that the determinant of the matrix of the four solutions is constant. Thus, we conclude that there is a constant K_P'

$$K_P' \leq |\det(y_1^P\ y_2^P\ y_3^P\ y_4^P)| \leq 1/K_P' \tag{A8}$$

uniformly in Im $s \leq 0$, $|s| \geq 1$. Here, as elsewhere in this proof, we must exclude a neighborhood of the origin, and consider domains lying in $|s| \geq 1$ only.

We prove that there is a solution $y_F(s)$ of (A1) defined in the half plane Im $s \leq 0$ such that for $-5\pi/6 \leq \arg s \leq 0$ y_F has the same asymptotic expansion as y_2, or for any P there is a constant $\bar{K}_P$

$$|y_F - y_2^P| \leq \bar{K}_P\, |\exp\{-is - i(b/2)\log s\}|/|s|^{P+1} \tag{A9}$$

while for $-\pi \leq \arg s \leq -5\pi/6$ y_F has the same asymptotic expansion as $y_2 + \lambda_P y_3$ or for any P there are constants λ_P and C_P such that

$$|y_F - y_2^P - \lambda_P y_3^P| \leq C_P\, |\exp\{-is - i(b/2)\log s\}|/|s|^{P+1}. \tag{A10}$$

It then follows that y_F represents a fast wave incident from the high field side (arg s = 0), that no fast wave reflected, and that the energy flux transmission coefficient is exp(-bπ). The mode converted amplitude λ_p cannot be determined explicitly.

The proof proceeds in four steps. First, we construct an equation approximately equal to (A1) with solutions y_1^P, y_2^P, y_3^P, y_4^P valid in the full domain Im s $\leq$ 0. We solve (A1) by perturbation about the approximate equation and we thus formulate an integral equation whose solutions solve (A1) and which are also close to our approximate equation. Secondly, we consider the integral equation in the sector $-5\pi/6 \leq \arg s \leq 0$ and show that it has solution, called y_F, which has as its asymptotic expansion y_2. Third, we examine the sector $-\pi \leq \arg s \leq -2\pi/3 - \delta$, $0 \leq \delta \leq \pi/6$, and we show that there are four solutions of the integral equation which are asymptotic to y_1, y_2, y_3, and y_4, respectively. The fourth, and the last, element of the proof is trivial given the first three steps. The solution y_F of step two and the four solutions of step three have a common domain of definition. Thus y_F is a linear combination of the four other solutions. Since y_1 and y_4 are exponentially large in the common domain y_F must be a linear combination of y_2 and y_3. When one matches the amplitudes of y_F, y_2, and y_3, one sees $y_F = y_2 + \lambda_p y_3$, and the proof is done. Thus, we prove only steps one, two, and three.

We start the proof with the construction of a comparison equation. We define

$$M^P = (y_1^P \; y_2^P \; y_3^P \; y_4^P), \tag{A11}$$

and we introduce

$$B^P = \left(\frac{d}{ds} M^P\right) (M^P)^{-1}. \tag{A12}$$

Since det M^P is limited by (A8) we know that B^P exists. Further

$$\frac{d}{ds} M^P = B^P M^P, \tag{A13}$$

so that M^P is a fundamental matrix of solutions for the equation

$$y' = B^P y. \tag{A14}$$

Further, by construction and the uniqueness of the asymptotic expansions we know that

$$|B^P - A| \leq C'_P/|s|^{Q_P} \tag{A15}$$

where both P and Q_P go to infinity together. We could determine the relation between P and Q_P, but it is of no significance in the proof. We may rewrite (A1) as

$$y' = B^P y + (A - B^P)y,$$

which can be written as an integral equation

$$y = y_0(s) + M^P(s) \int^s [M^P(s')]^{-1} [A(s') - B(s')] y(s') ds', \tag{A16}$$

where y_0 is any solution of (A14), that is a linear combination of y_1^P, y_2^P, y_3^P, and y_4^P.

In order to construct solutions of (A16) we must specify the lower limits of the integration in (A16), specify $y_0(s)$, and then demonstrate that a unique solution of (A16) exists. We would then also like to show that the asymptotic expansion of this solution is, up to some order, $y_0(s)$. We will only sketch the proof which is, in every case, a straightforward application of Picard iteration. Picard iteration will also show that the asymptotic expansion of y(s) is, up to some order $y_0(s)$. In order to describe the

proof further it is convenient to introduce the notation for the exponential factors in (A4)-(A7):

$$z_1 = \exp[is + i(b/2)\log s] = 1/z_2 \tag{A17}$$

$$z_3 = \exp(cs^{3/2} + d\, s^{1/2}) = 1/z_4. \tag{A18}$$

Then, we may identify the structure of $M_P(s)\,[M_P(s')]^{-1}$ as

$$\begin{aligned} M_P(s)\,[M_P(s')]^{-1} &= (s')^{-1/2}[z_1(s)z_2(s')\,N^P_{12}(s,s') + z_2(s)z_1(s')\,N^P_{21}(s,s')] \\ &+ (ss')^{-1/4}[z_3(s)z_4(s')\,N^P_{34}(s,s') + z_4(s)z_3(s')\,N^P_{43}(s,s')], \end{aligned} \tag{A19}$$

where the four by four matrices N^P_{ij} are all bounded in Im $s \leq 0$ and Im $s' \leq 0$, $|s| \geq 1$, $|s'| \geq 1$, so that

$$|N^P_{ij}(s,s')| \leq L_P. \tag{A20}$$

We now carry out the second step of the proof. We restrict ourselves to the sector $-5\pi/6 \leq \arg s \leq 0$, and we set y_0 in the integral equation equal to $y^P_2(s)$. We select different end points of integration for the four types of terms that appear in the kernel as shown in (A19). We expect to show that $y \sim y^P_2$. Thus, for the first two types of terms $z_1(s)\,z_2(s')$ and $z_2(s)\,z_1(s')$ we take the path of integration to be the line Re s' = constant. On this path, provided $y \sim y^P_2 \sim z_2$ the exponential factors take their largest values at the end point $s' = s$, and we may take these exponential factors outside the integral, evaluated at $s' = s$. Now z_3 is exponentially large in the sector $-\pi/3 < \arg s \leq 0$ and exponentially small in the sector $-\pi < \arg s < -\pi/3$. Where z_3 is large z_4 is small, and conversely. The integrals containing $z_3(s')$ start at infinity in the sector where z_3 is small, and correspondingly for $z_4(s')$. It is again easy to adjust the paths of integration such that the exponential factors are largest at $s' = s$, so

that these factors can be taken out of the integral. We now define a norm of a function

$$\| y\|_R = \sup_{\substack{|s|\geq R \\ -5\pi/6 \leq \arg s \leq 0}} \| y(s)\ z_1(s)\| . \tag{A21}$$

It then easily follows that

$$\| M^P(s) \int^s [M^P(s')]^{-1}[A(s')-B^P(s')]\ y(s')\ ds'\| \leq \frac{16\ L_P}{(Q_P-3/2)} R^{-Q_P+3/2} \tag{A22}$$

Thus, for any P sufficiently large that $Q_P > 3/2$, there is a radius R such that the integral operator defines a contractive mapping for $|s| > R$ in the sector $-5\pi/6 \leq \arg s \leq 0$. Hence the integral equation has a unique solution in functions of bounded norm (A21). This solution satisfies the original equation (A1), and may be extended uniquely to the full lower half plane $\text{Im } s \leq 0$. Further, the estimate (A22) also shows that to leading order (actually order $Q_P - 3/2$) the solution has the same asymptotic expansion as y_2^P. It is this solution we call y_F.

We do not give the third step of the proof as it is a repetition of the above argument in the sector $-\pi \leq \arg s \leq -2\pi/3 - \delta$, $0 \leq \delta \leq \pi/6$. One replaces y_0 in (A16) successively by y_1^P, y_2^P, y_3^P, and y_4^P, and adjusts the paths of integration to infinity such that the exponential factors in the integrands are the largest at the end point $s' = s$. An alternative proof of the validity of the expansions in this sector, in which no expansions change from large to small, or vice versa, is given in Coddington and Levinson. The proof is, in any case, quite straightforward, and we omit it.

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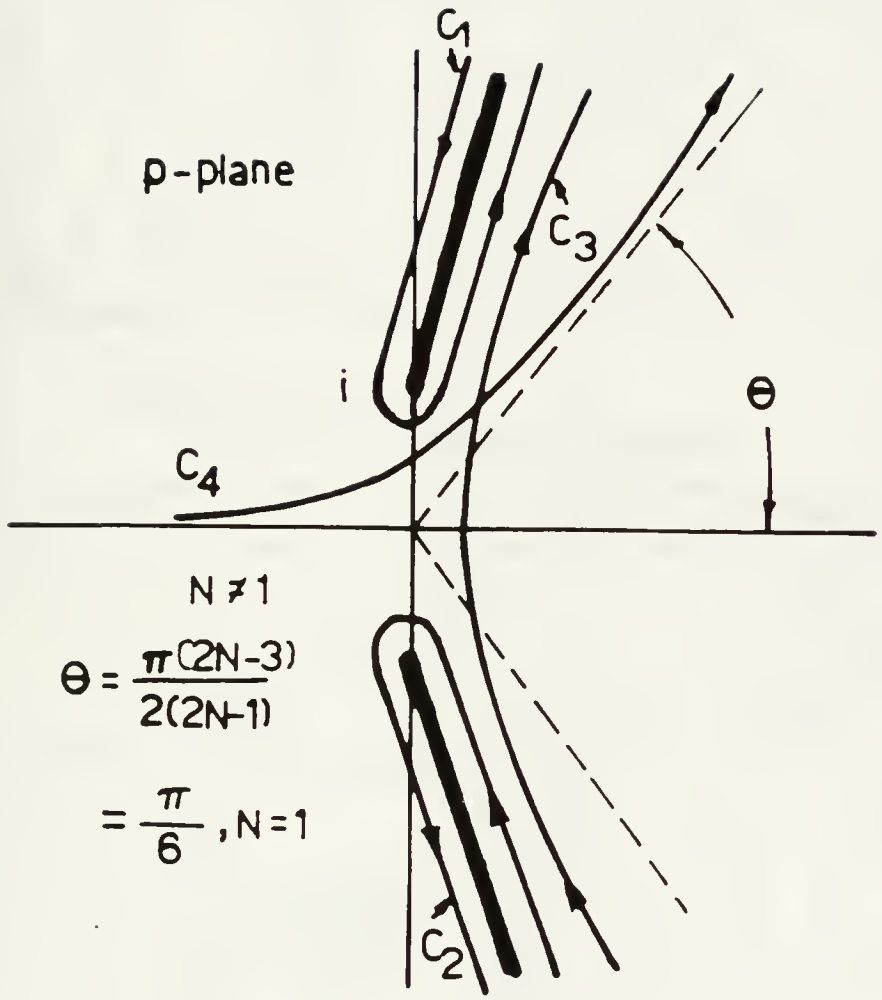

Fig.1 Contours C_1, C_2, and C_3 in the p-plane which generate the solutions u_+, u_-, and u_3, respectively, of Eq.(27).

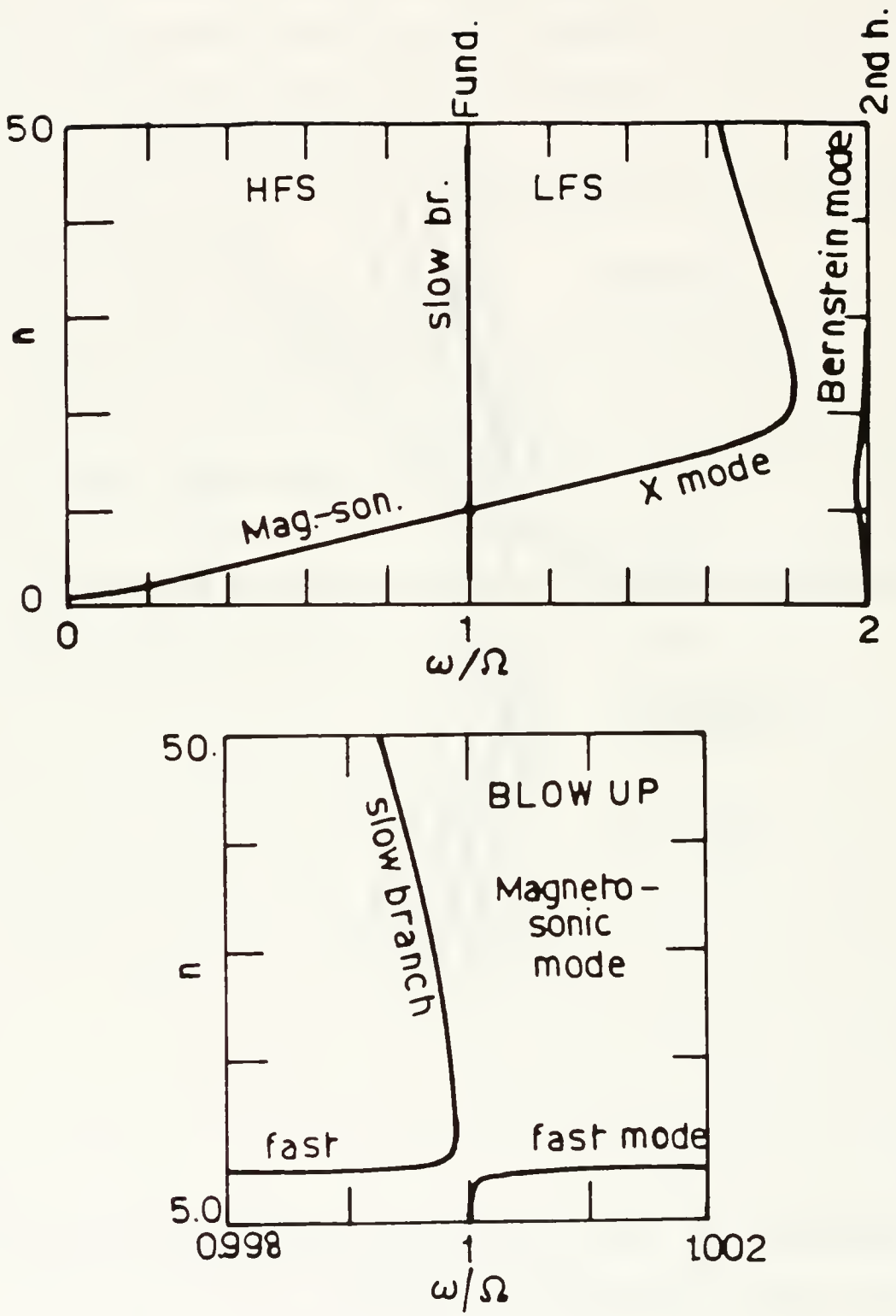

Fig.2 Uniform field dispersion relation for the magnotesonic waves near the fundamental ion resonance.

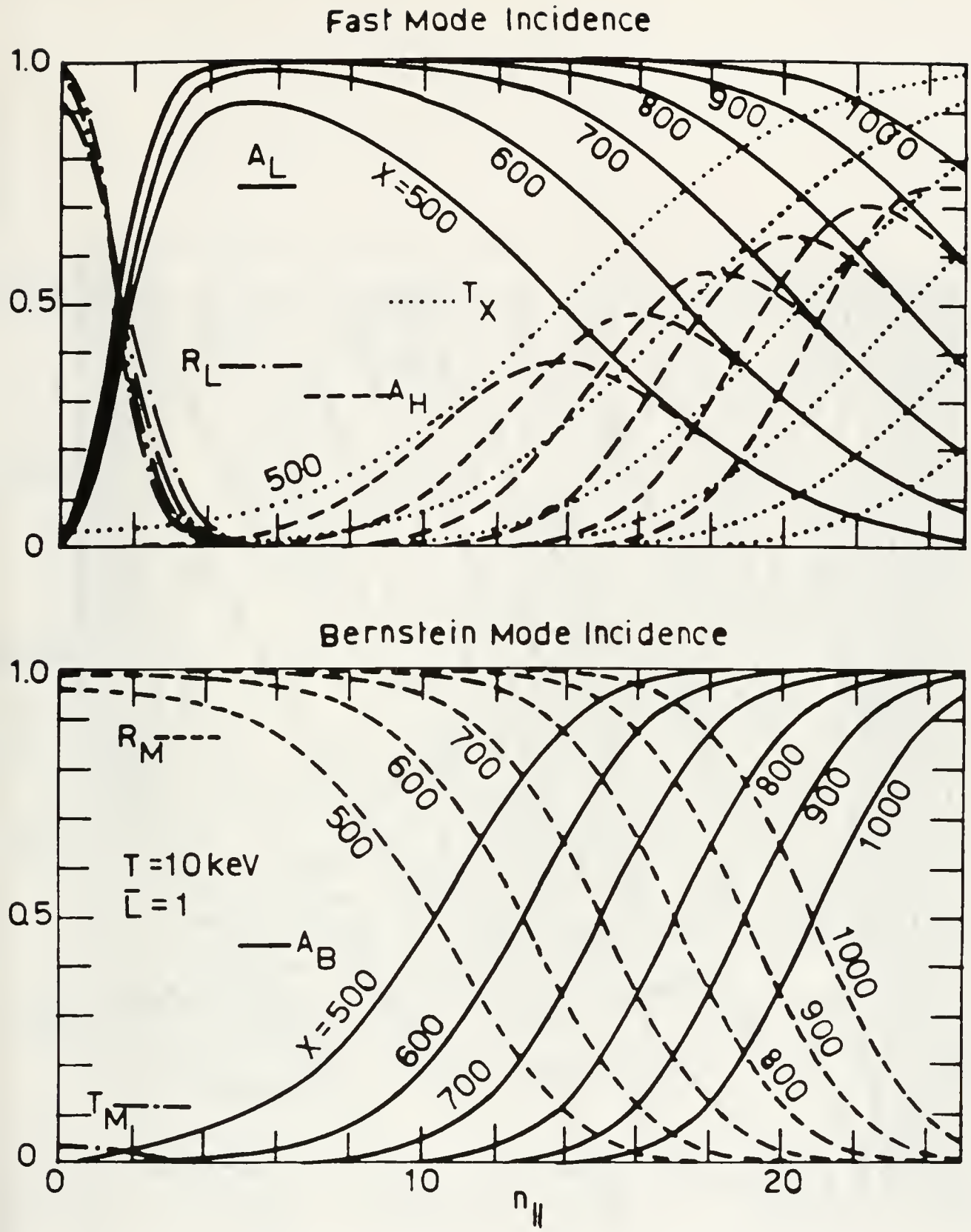

Fig. 3 Propagation characteristics for waves across the third ion cyclotron resonance harmonic for T_H = 10 keV, and for various values of X_H and $\bar{L} = L\omega/c = 1$, as a function of $n_{||}$. A_L, A_H, A_B denote the absorption coefficients for LFI, HFI, and Bernstein mode incidence, respectively. R_L, and R_B are the reflection coefficients for LFI, and BMI, R_M and T_M denote the reflected and transmitted mode conversion coefficients.

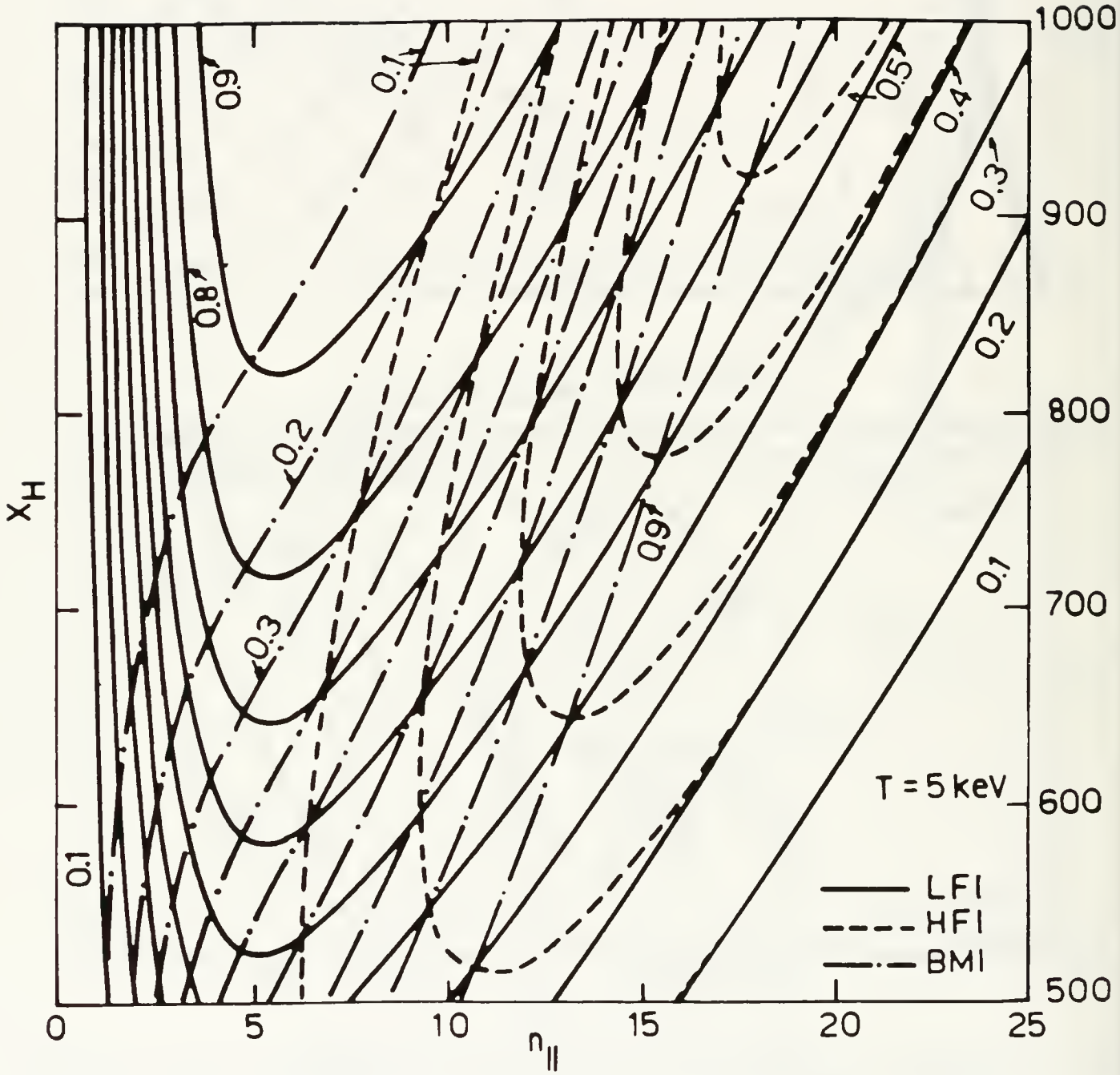

Fig.4 Absorption contours for the third harmonic resonance for T = 5 keV and $\bar{L}$ = 1 in the (X_H,$n_{||}$) plane.

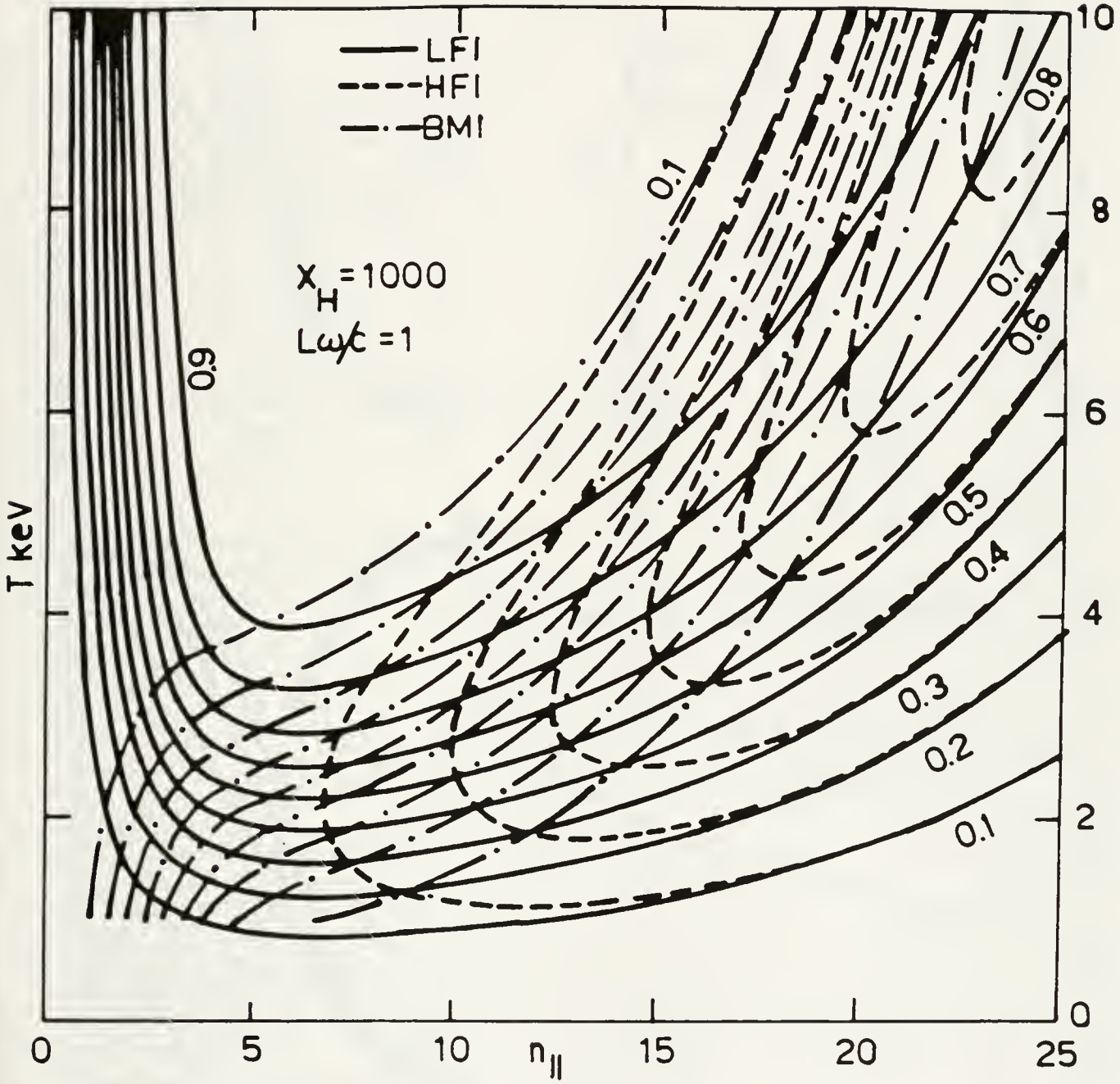

Fig.5 Absorption contours for the third harmonic resonance for $X_H = 1000$ and $\bar{L} = 1$ in the $(T, n_{||})$ plane.

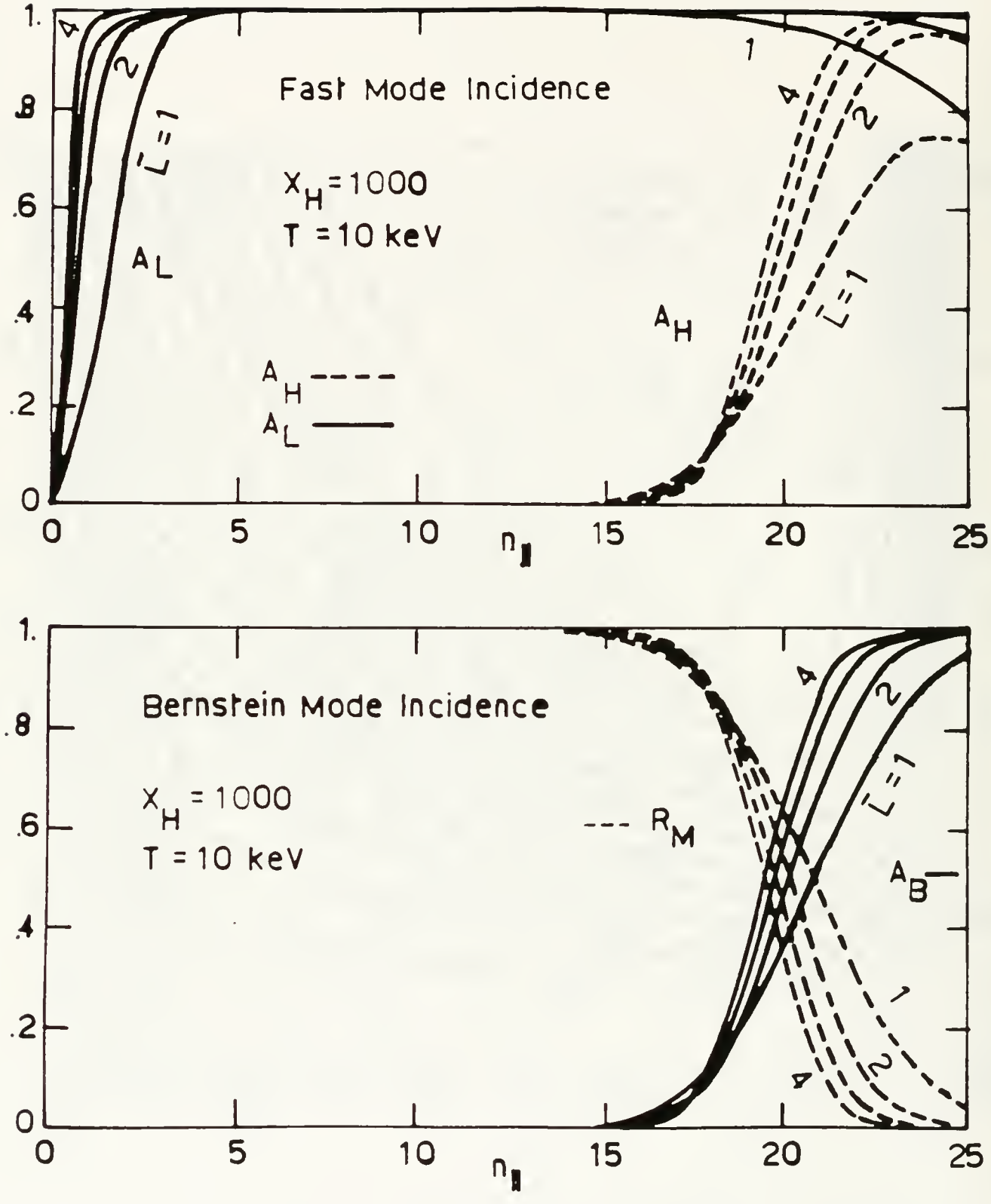

Fig.6 The effect of the machine size, or $\bar{L}$, on the absorption coefficient, for X_H = 1000, T = 10 keV, for the third harmonic resonance.

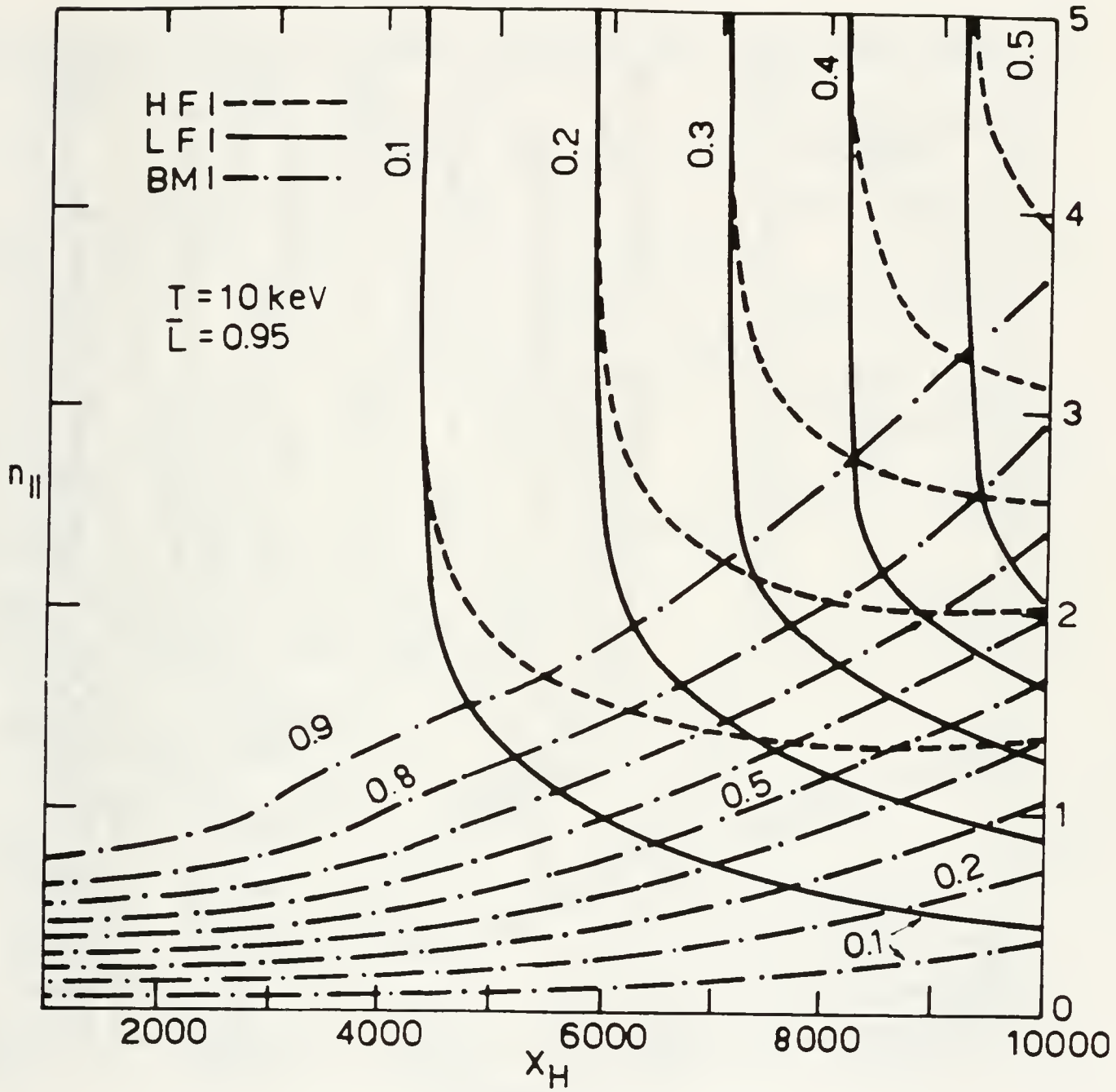

Fig.7 Absorption contours for T = 10 keV, $\bar{L}$ = .95, in the $(X_H, n_{||})$ plane for magnetosonic waves across the fundamental ion resonance layer.

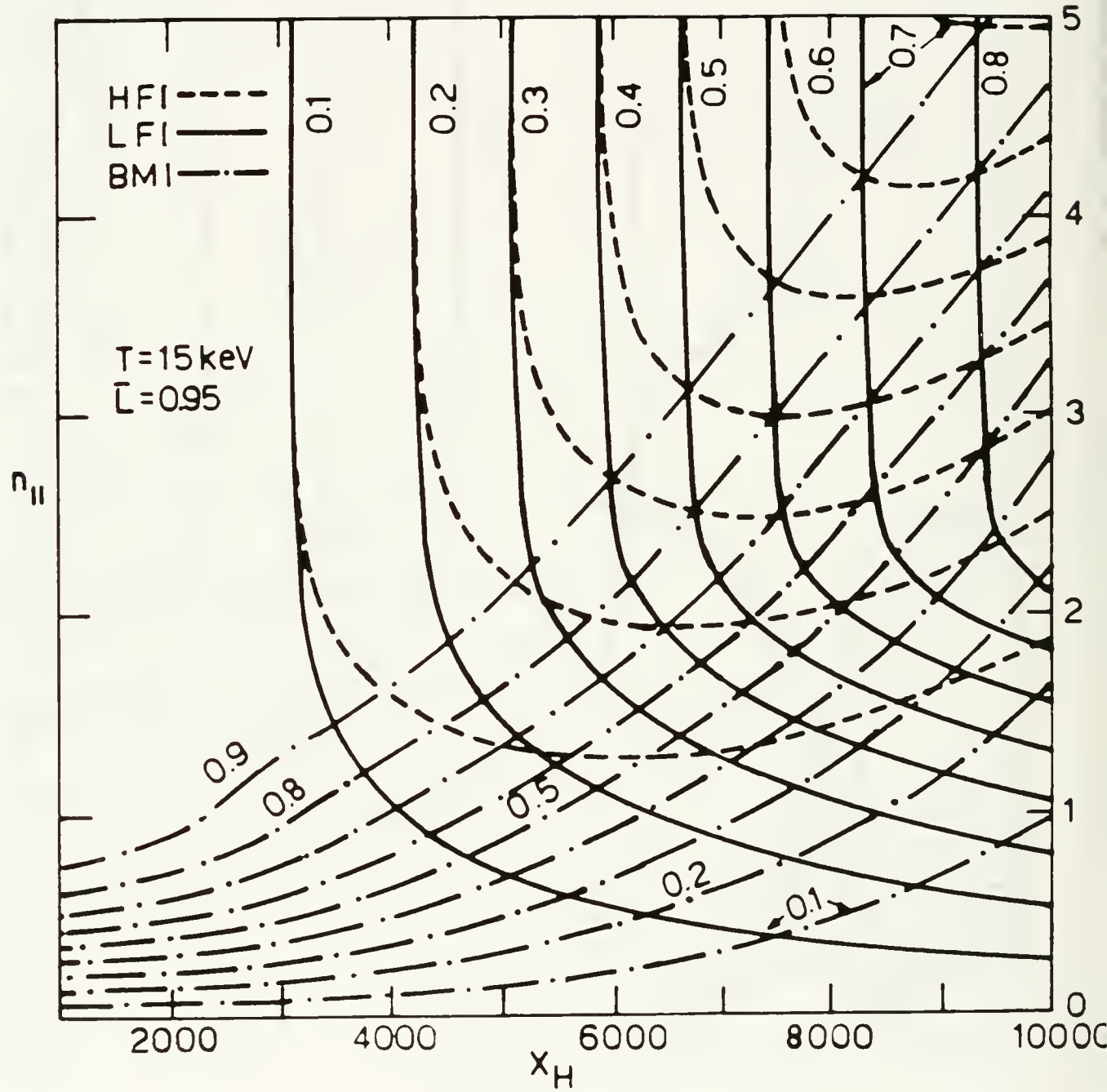

Fig.8 Absorption contours for T = 15 keV, $\bar{L}$ = .95, in the $(X_H, n_{||})$ plane for magnetosonic waves across the fundamental ion resonance layer.

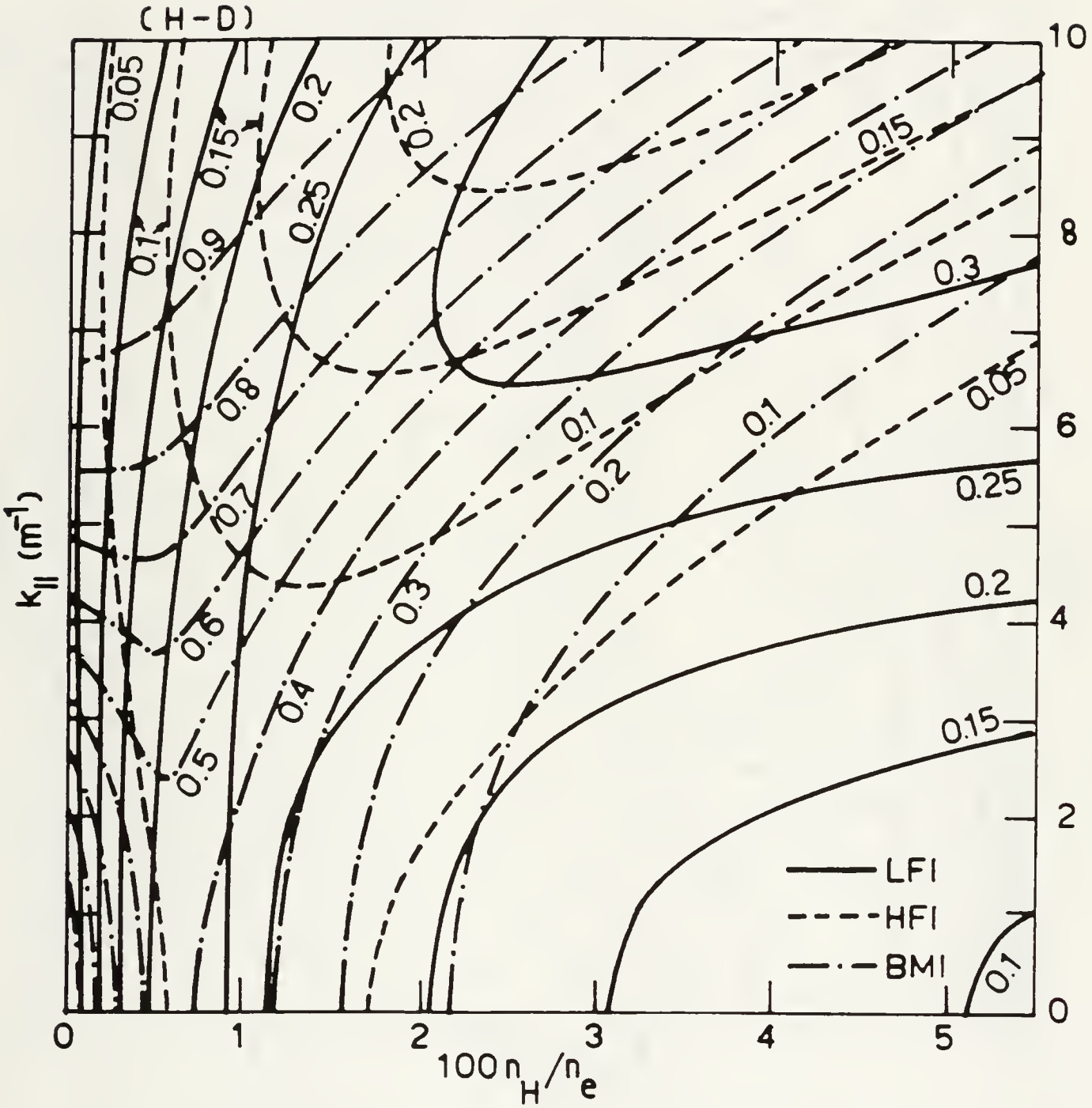

Fig.9 Absorption contours for the ion-ion hybrid resonance in the minority density and parallel wavenumber plane for a PLT example with B_0 = 3T, L = 130 cm, $T_D = T_H$ = 2 keV, $n_e = 3 \times 10^{13} cm^{-3}$.

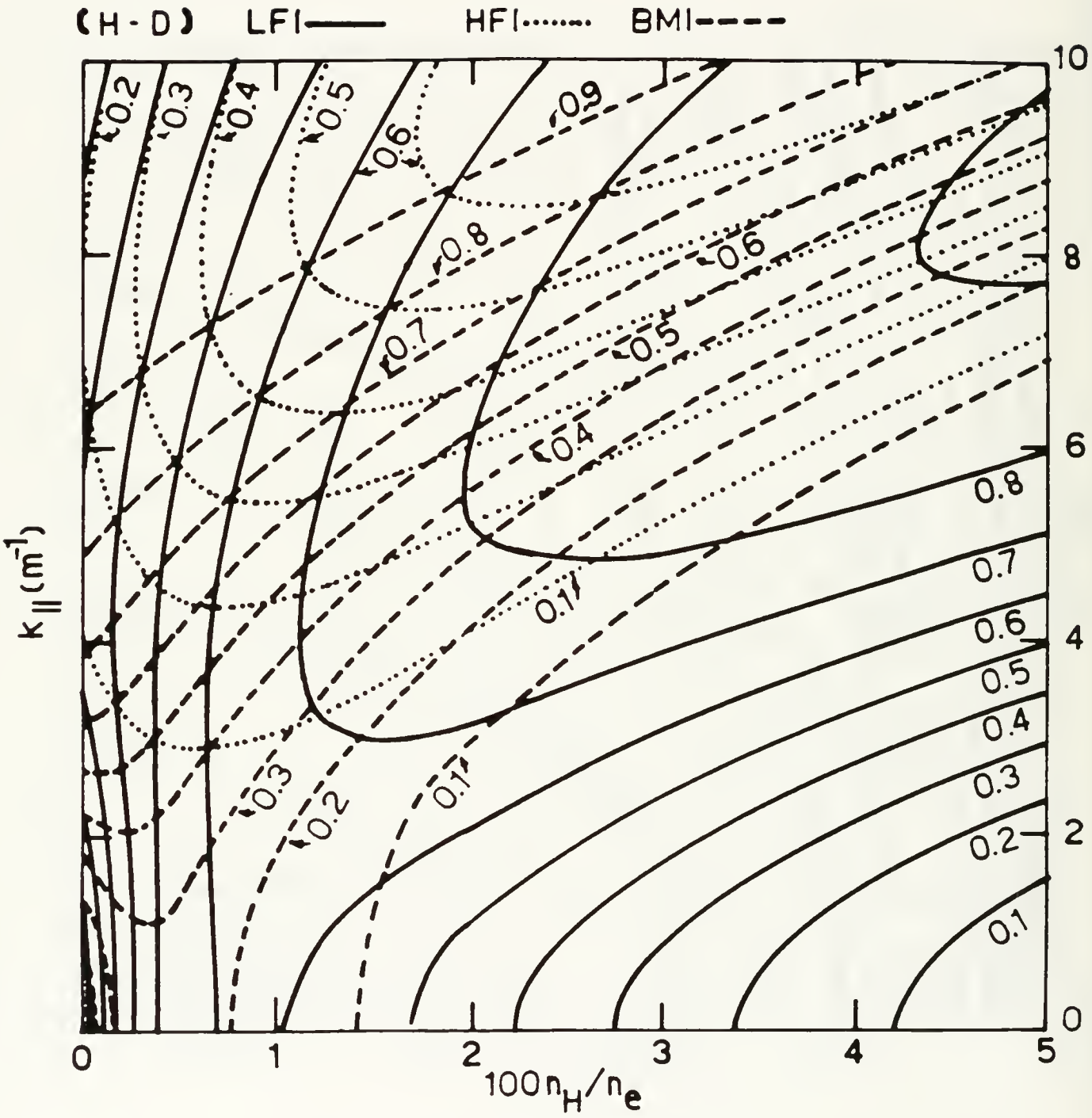

Fig.10 The same as Fig.9 but for a JET example with B_0 = 3.45T, L = 3m, T_D = T_H = 5 keV, n_e = 3.5 $\times 10^{13} cm^{-3}$.

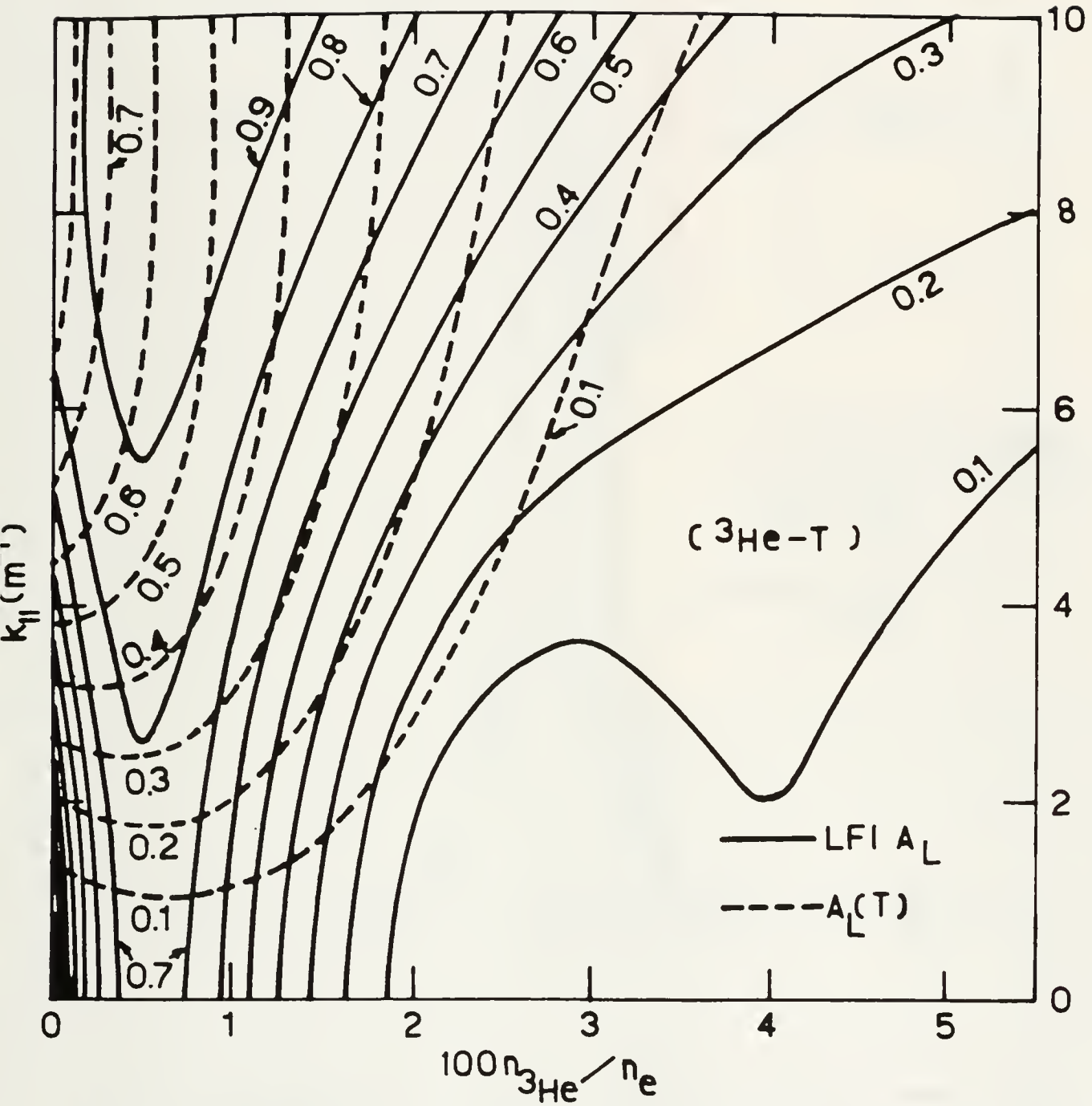

Fig.11 The same as Fig.9 but for a CIT example with B_0 = 10 T, L = 122 cm, $T_{He} = T_T$ = 10 keV, $n_e = 5 \times 10^{14} cm^{-3}$. Dashed lines denote level lines for the energy absorption by tritium for the low field side incident fast mode.

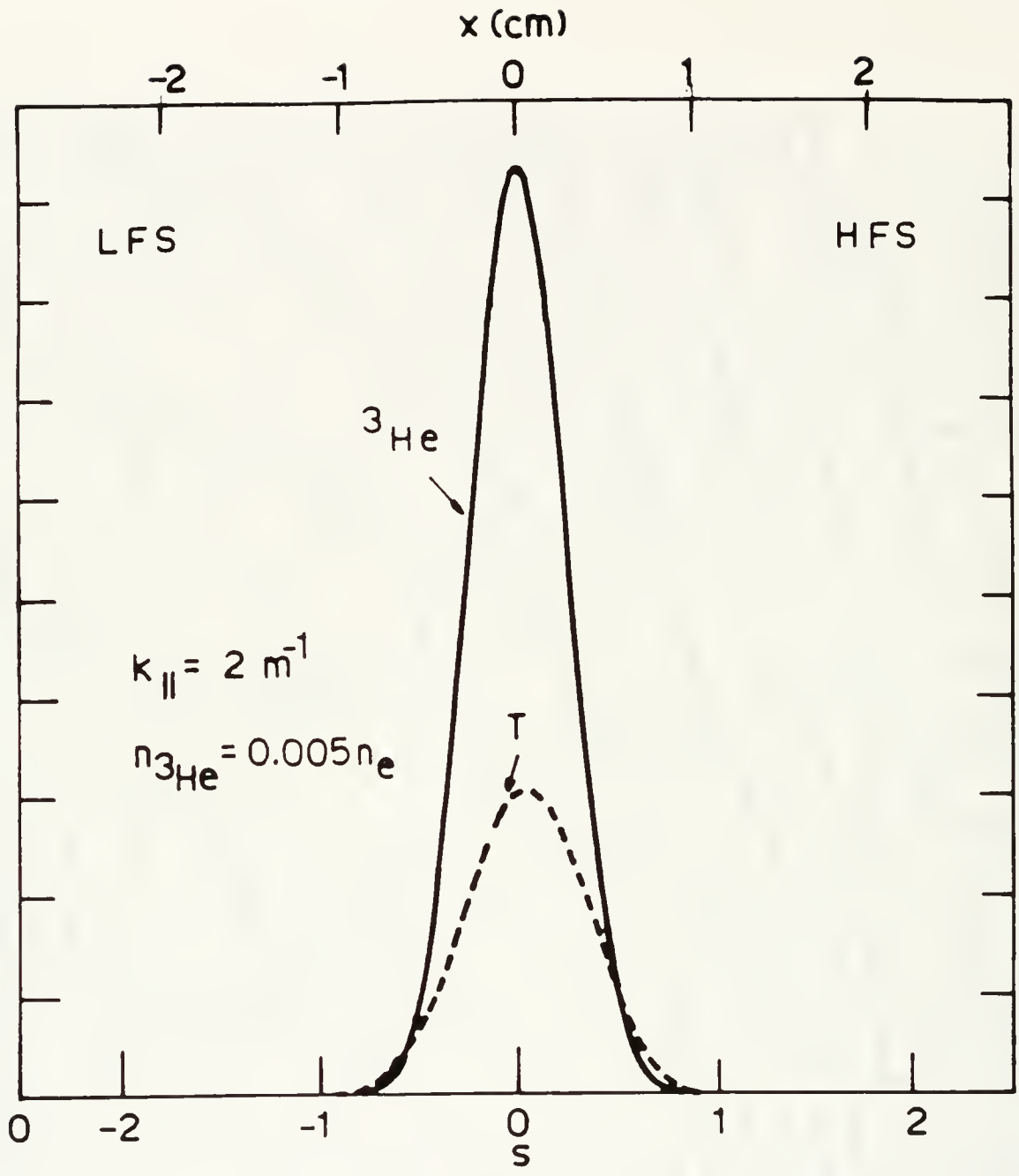

Fig.12 Power deposition profiles for ^{3}He and T in the CIT example of Fig.11, for $n_{He} = 0.005\ n_e$ and $k_{||} = 2$ 1/m.

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